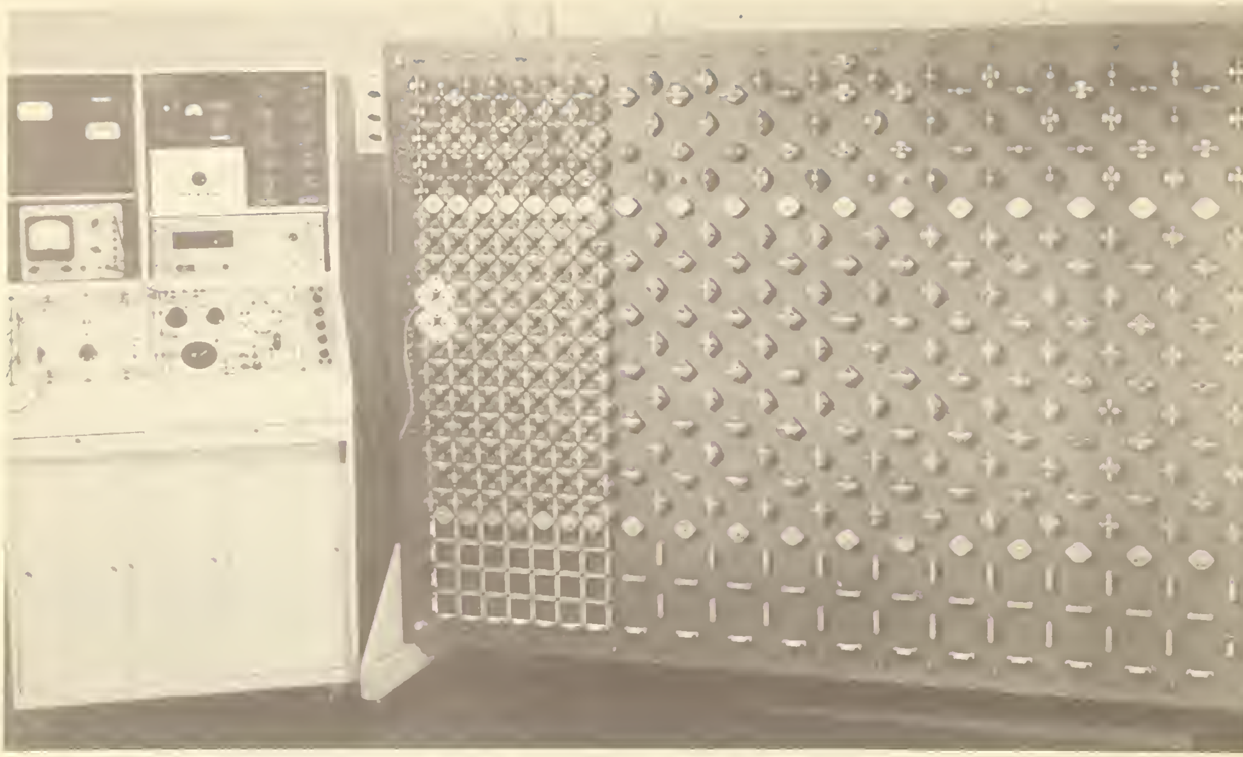
# SIMULATING WATER FLOW IN SOIL

## with an electrical resistance network



OCTOBER 1962 ARS 41-65
Agricultural Research Service
UNITED STATES DEPARTMENT OF AGRICULTURE



An electrical resistance network (right) and its control console (left) used in studying steady-state flow of water in soil. The network consists of 575 resistive units, arranged in a fine mesh on one-fourth of the 6- by 8-foot board. The console contains electrical equipment for supplying and measuring electrical voltages and currents in the network. In this photograph the network is assembled to simulate water flow into a drain that is buried in a 3-layered soil. The seven brightly colored resistors on the left represent the drain and can be adjusted to simulate drains of different sizes.

Prepared in

Soil and Water Conservation Research Division
Agricultural Research Service
United States Department of Agriculture

in cooperation with

Ohio Agricultural Experiment Station

# CONTENTS

# SIMULATING WATER FLOW IN SOIL WITH AN ELECTRICAL RESISTANCE NETWORK [1]

By Bunyut S. Vimoke and George S. Taylor[2]

Water infiltration, redistribution, and removal from soil are important processes that affect all land management systems. Although these processes must be studied under field and laboratory conditions, certain phases can be evaluated by analog techniques with a considerable saving in time and cost and in accuracy. Electrical analogs have been very useful in these evaluations. Two types have been used with good success: (1) those utilizing the flow of electricity through sheets of electrical conducting paper and (2) those using a network of electrical resistors. Electrical conducting paper can be used to solve steady-state flow problems in porous media that are saturated, isotropic, and homogeneous with respect to its hydraulic conductivity. An electrical resistance network is more flexible in its use than conducting paper. The network can quickly be adjusted to simulate flow conditions in soils that are homogeneous or stratified, saturated or unsaturated, isotropic or anisotropic. Various investigators have already utilized the network to study flow of water either in homogeneous or stratified medium (1, 6, 7, 8).[3]

In this report, the major objective is to present a method for calculating and assembling network resistances to represent water flow in soil. The approach used herein is called the "building block" method. The soil profile is considered to be composed of discrete, rectangular blocks of soil. These blocks are then joined to "build" the entire soil profile. Each block of soil is represented by a mesh of four resistors. The meshes are then joined to form the network, which, in turn, represents the soil profile. The discussion in this report will deal with representations of steady-state flow of fluid in saturated and unsaturated media. Procedures will be given for representing media that are either homogeneous or stratified and either isotropic or anisotropic with respect to soil hydraulic conductivity. Formulas will be given for calculating resistances when rectangular blocks of different dimensions are used. Special formulas are presented for calculating resistances adjacent to a tubular drain embedded in the soil.

---

[1]Cooperative investigations conducted under cooperative agreement between the Ohio Agricultural Experiment Station and the Soil and Water Conservation Research Division, Agricultural Research Service, U.S. Department of Agriculture.

[2]Bunyut S. Vimoke, formerly Research Assistant, Ohio State University, Columbus; George S. Taylor, Professor of Agronomy, The Ohio State University, Columbus, and collaborator, Soil and Water Conservation Research Division, Agricultural Research Service, U.S. Department of Agriculture.

[3]Underscored figures in parentheses refer to Literature Cited at end of report.

## CORRESPONDENCE BETWEEN ELECTRICAL AND WATER FLOW IN A CONDUCTING MEDIUM

The equations for the potential function $\phi$ (9) of fluid flow through a saturated, isotropic medium under steady-state conditions satisfy Laplace's equation, namely, $\nabla^2\phi = 0$. Various investigators (4; 9) have utilized numerical solutions to solve Laplace's equation for two dimensional flow into a buried drain. If square meshes are drawn as shown in figure 1 and values of hydraulic head $\phi$ are assigned to each corner, it can be shown by the relaxation method (9) that for a uniform medium:

$$\phi_0 = (\phi_1 + \phi_2 + \phi_3 + \phi_4)/4 \tag{1}$$

The relaxation method can be applied to an analogous problem in electrical circuit theory: If one has a network of equal resistances, the potential $\underline{V}_0$ at a node is equal to one-fourth the sum of those at the four adjacent nodes. That is,

$$\underline{V}_0 = (\underline{V}_1 + \underline{V}_2 + \underline{V}_3 + \underline{V}_4)/4 \tag{2}$$

This can be proved by utilizing Kirchhoff's and Ohm's Laws. If Kirchhoff's Law is applied to figure 2, the algebraic sum of the currents $\underline{i}$ entering and leaving the node of potential $\underline{V}_0$ is equal to zero. That is --

$$\underline{i}_1 + \underline{i}_2 + \underline{i}_3 + \underline{i}_4 = 0 \tag{3}$$

If Ohm's law is used, equation 3 becomes --

$$\frac{(\underline{V}_1 - \underline{V}_0)}{\underline{R}_1} + \frac{(\underline{V}_2 - \underline{V}_0)}{\underline{R}_2} + \frac{(\underline{V}_3 - \underline{V}_0)}{\underline{R}_3} + \frac{(\underline{V}_4 - \underline{V}_0)}{\underline{R}_4} = 0 \tag{4}$$

Solving equation 4 for $\underline{V}_0$, one obtains the following:

$$\underline{V}_0 = \frac{\underline{V}_1/\underline{R}_1 + \underline{V}_2/\underline{R}_2 + \underline{V}_3/\underline{R}_3 + \underline{V}_4/\underline{R}_4}{1/\underline{R}_1 + 1/\underline{R}_2 + 1/\underline{R}_3 + 1/\underline{R}_4} \tag{5}$$

In the special case where all of the values of resistance are equal, i.e., $\underline{R}_1 = \underline{R}_2 = \underline{R}_3 = \underline{R}_4 = \underline{R}_0$ (square network representing uniform media), the potential $\underline{V}_0$ will be given by equation 2. Thus the voltage $\underline{V}_0$ is analogous to the potential $\phi_0$ used in the relaxation method.

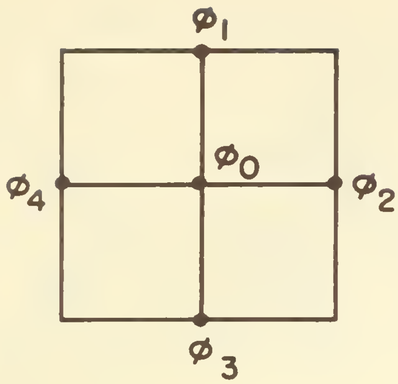



FIGURE 1.—Hydraulic head potential at corners of a square grid.

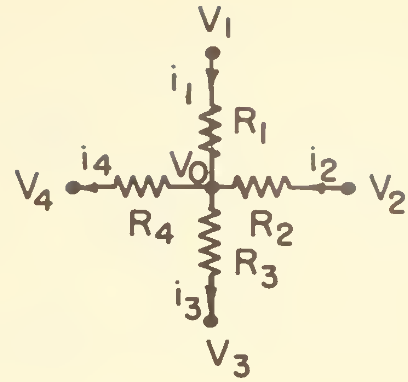



FIGURE 2.—Portion of an electrical resistance network.

## GENERAL CONCEPT OF A RESISTANCE NETWORK

### Representing Soil Medium by Electrical Conducting Paper

If one has a square piece of conducting paper with dimension $\underline{a}$ by $\underline{a}$, it can be used to represent any square dimension of uniform soil medium having unit thickness (see appendix, part I). To illustrate, first select a piece of Teledeltos paper[4] of dimension 1 inch by 1 inch. This piece of paper can represent a block of soil medium one foot in the horizontal direction and one foot in the vertical direction. The same piece of paper can also be used to represent a 10 by 10-foot block of soil. In both cases, the block of soil is of unit thickness. For example, the block of soil represented in the first case has the dimensions of a cube of one foot, and the block represented in the second case has the dimensions of 10 by 10 feet by 1 foot. A piece of conducting paper of dimension 5 by 5 inches will serve the same purpose.

### Representing Soil Medium by Network of Resistors

Consider the boundaries formed by the four lines that bound the square of resistive paper shown in figure 3a. If we are not concerned with conditions that exist inside these boundaries, we would obtain the same information from using either resistive paper or a group of resistors. It will now be shown that four resistors can be used to represent each square of resistive paper. Since

---

[4]A commercial name for electrical-conducting paper. This type of paper is often called "resistive" paper. The use of this or other commercial products in this report does not imply approval of the product to the exclusion of others that may also be suitable.

a block of uniform soil can be represented by a square of resistive paper, it will also follow that the four resistors will do likewise.

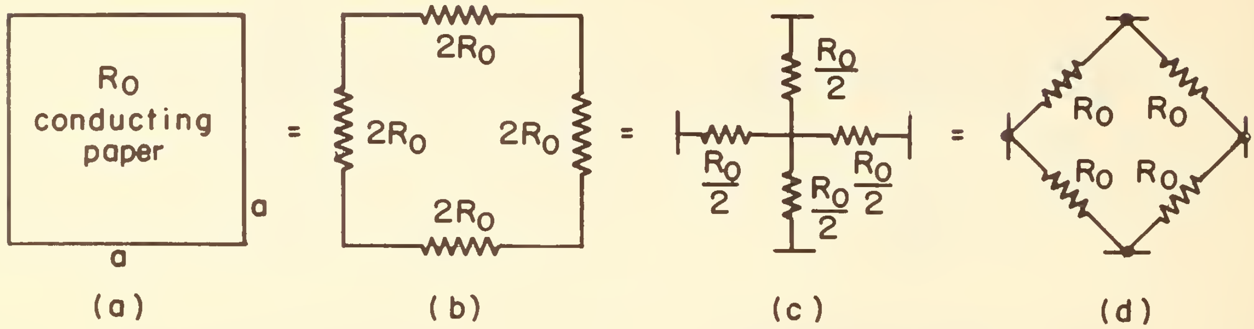



FIGURE 3.—Ways to represent a square piece of electrical conducting paper by a group of resistors.

If groups of four resistors are used to represent a square block of soil, there are different ways to put them together (see fig. 3). The arrangement that is most convenient is shown in figure 3, b, and this arrangement of resistors will be used throughout the rest of the discussion.

The justification of using four resistors to represent a square piece of resistive paper can be shown as follows (figs. 4 and 5). Take a square of resistive paper and paint two highly conductive lines on it (fig. 4). Silver paint is usually used, similar to that used in printed circuits. Measure the electrical resistance $\underline{R}_o$ between these lines. The resistance $\underline{R}_o$ is called the "characteristic resistance" of the medium, and its reciprocal yields the electrical conductivity $\underline{G}$. Now do the same with the group of four resistors, using wires instead of painted lines, which is electrically the same thing (fig. 5). In all cases, the resistance is $\underline{R}_o$ ohms, which is the characteristic resistance of the resistive paper.

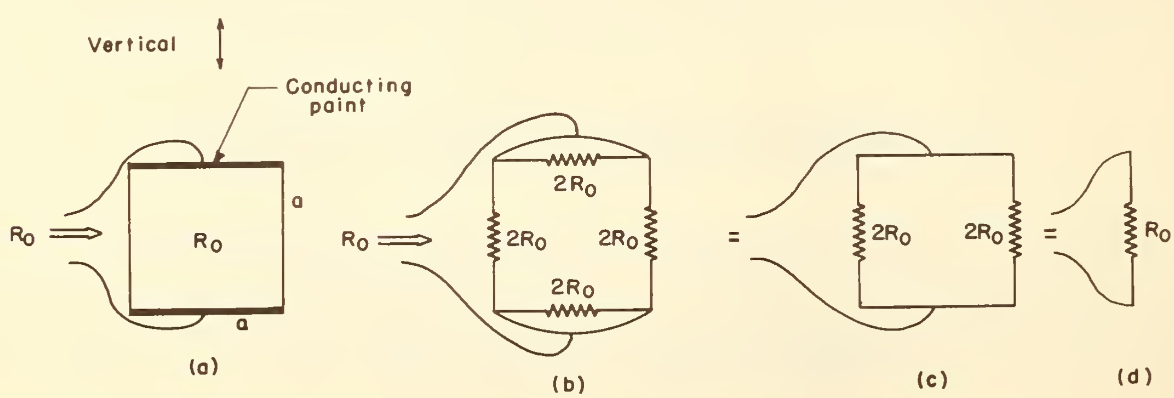



FIGURE 4.—Square soil block or a square of conducting paper with two highly conductive lines painted on it to verify use of four resistors.

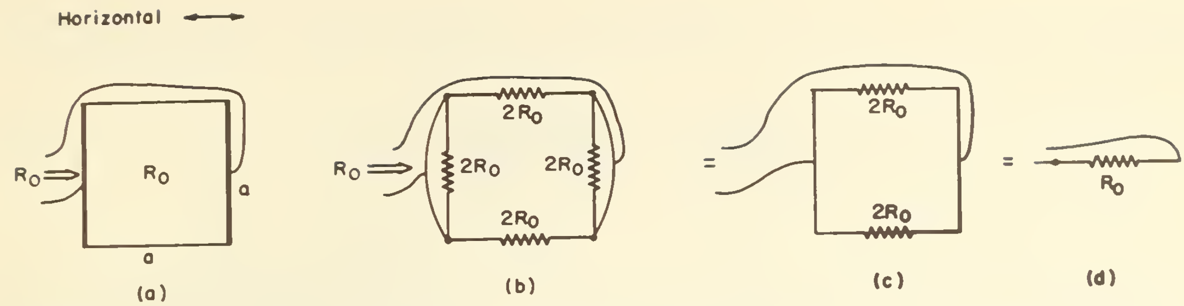



FIGURE 5.—Square soil block or a square of conducting paper on which wires are used instead of painted lines in figure 4, to verify use of four resistors.

The directions in which the resistances are measured are orthogonal. For a two-dimensional flow problem, we can specify the resistance of a part of the medium by the resistances measured in just two directions. The resistance of the medium in any other direction can be expressed in terms of its horizontal and vertical components. For convenience, one of these directions is chosen to be the vertical, and the other, horizontal. This choice is made because many boundaries in water-flow problems are usually either horizontal or vertical and because soils are principally anisotropic with respect to hydraulic conductivity in these two directions.

In regard to representing a cross-sectional area of soil with four resistors, the same conclusion can be drawn here as with the square of conducting paper. The mesh of resistors shown in figure 3, b can represent a square block of soil of any dimension. It is true, however, that more information can be obtained from a resistance network if the blocks of soil represented in this manner are small. This is because one cannot make electrical measurements inside the simulated block of soil.

## THE "BUILDING BLOCK" APPROACH OF REPRESENTING SOIL MEDIUM

Two resistors in parallel can always be combined into a single resistor. The use of a single resistor is merely an economical convenience. Two resistors ($\underline{R}_1$ and $\underline{R}_2$) in parallel are equivalent to a single resistor having a resistance of $\underline{R}$ ohms (fig. 6).

$$\underline{R}_1 /\!/ \underline{R}_2 = \underline{R}$$

$$\frac{1}{\underline{R}} = \frac{1}{\underline{R}_1} + \frac{1}{\underline{R}_2} \qquad (6)$$

$$\underline{R} = \frac{\underline{R}_1 \underline{R}_2}{\underline{R}_1 + \underline{R}_2}$$

If both $\underline{R}_1$ and $\underline{R}_2$ are equal to $2\underline{R}_o$ ohms, then $\underline{R}$ is equal to $\underline{R}_o$ as shown by equation 6a.

$$\underline{R} = \frac{\underline{R}_1 \underline{R}_2}{\underline{R}_1 + \underline{R}_2} = \frac{2\underline{R}_o \cdot 2\underline{R}_o}{2\underline{R}_o + 2\underline{R}_o} = \underline{R}_o \tag{6a}$$

In most cases, the magnitude of a single resistor in the network is derived from a consideration of two resistance values originally being in parallel. In assembling a network it is desirable to think in terms of the original resistance values that compose the building block[5] rather than the values of single resistors that result from parallel combinations. To illustrate further, look at the square grid ABCD (fig. 7, c with resistors $\underline{R}_o$). Actually, this is just the end result when the building blocks of figure 3, b have been joined. In the assembled network, all parallel resistors have been combined into single resistors. It is important to remember that we started with our basic building blocks as shown in figure 7, a. If there had been no other resistors in an adjacent mesh to form a parallel combination, we would have left it unaltered; and its value would have remained at $2\underline{R}_o$. This is what happens at the boundaries of a resistance network (see fig. 8). The resistors along the top surface and down the right side had no others with which to combine, so they were left at their original value of $2\underline{R}_o$. The reasoning is clear from the building block approach why boundary resistors are always twice the value of interior resistors in a resistance network.

## REPRESENTING A RECTANGULAR BLOCK OF SOIL

Up to this time we have only considered square building blocks, these being a specialized case of the more general rectangular shape. To determine the value of the four resistors that represent a rectangular block, we will introduce a slightly modified approach: Consider a rectangular block of soil and a piece of resistive paper of the same shape. This block of soil can also be represented by four resistors as shown in figure 9. The problem now is to determine the values of $\underline{R}_a$ and $\underline{R}_b$. To determine $\underline{R}_a$, visualize the procedure outlined in figure 10. A piece of resistive paper is cut horizontally into two equal sections. Each of the sections of figure 10, b can be thought of as being represented by a resistor of value $\underline{R}_a$ (figs. 10, c, 10, d, and 10, e). This value can be computed by equation 7:

$$\underline{R}_a = \frac{\text{length}}{\text{width}} \underline{R}_o = \frac{\underline{a}}{\underline{b}/2} \underline{R}_o = \frac{2\underline{a}}{\underline{b}} \underline{R}_o \tag{7}$$

The total resistance in the horizontal direction is given by equation 8.

$$\underline{R}_H = \underline{R}_a /\!/ \underline{R}_a = \frac{\underline{R}_a \cdot \underline{R}_a}{\underline{R}_a + \underline{R}_a} = \frac{\underline{R}_a}{2} = \frac{\underline{a}}{\underline{b}} \underline{R}_o \tag{8}$$

---

[5]In assembling the network, the resistors representing individual blocks of soil are joined to form the entire network. We choose to call this individual block of soil and the mesh of four resistors that represent it a "building block."

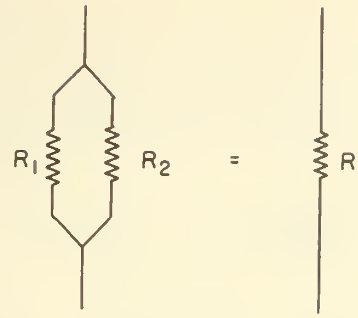



FIGURE 6.--Joining two resistors in parallel.

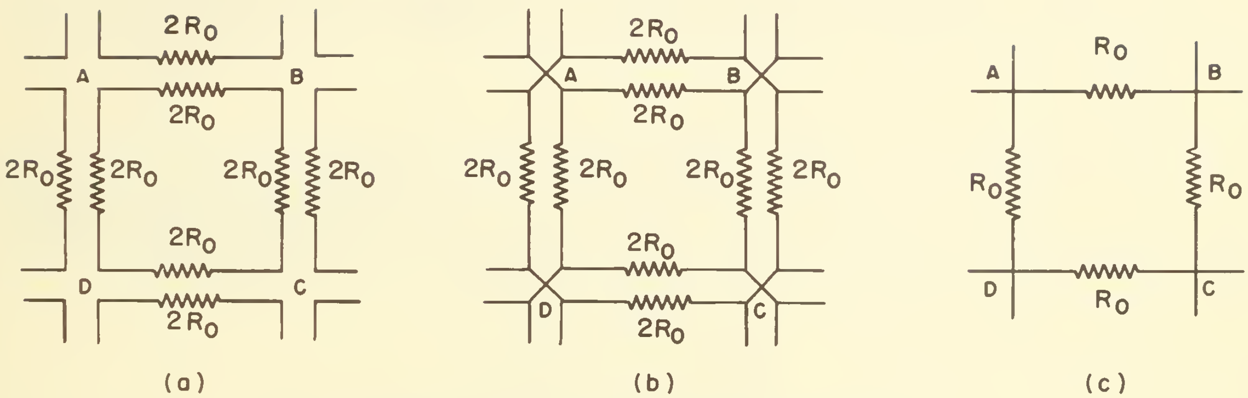



FIGURE 7.--Joining square-grid building blocks.

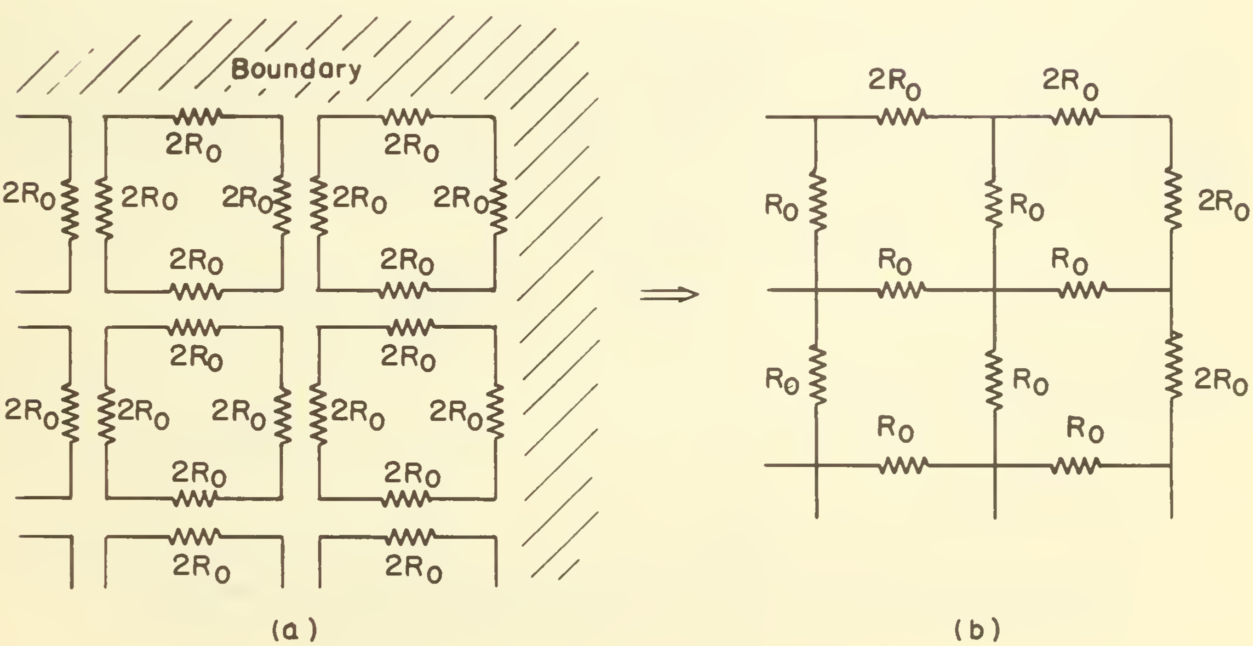



FIGURE 8.—Network of resistors, showing the value of the resistance along the boundaries.

To obtain the values for the vertical resistors, the same approach is used. By taking a piece of resistive paper representing our block of soil as shown in figure 11, this time we cut it in the vertical direction. Each of the two resulting equal parts can be thought of as being represented by a resistor of $\underline{R}_b$ ohms. If we then utilize equations 9 and 10 the total resistance in the vertical direction $\underline{R}_V$ can be obtained.

$$\underline{R}_b = \frac{\text{length}}{\text{width}} \underline{R}_0 = \frac{b}{a/2} \underline{R}_0 = \frac{2b}{a} \underline{R}_0 \tag{9}$$

$$\underline{R}_V = \underline{R}_b // \underline{R}_b = \frac{\underline{R}_b \cdot \underline{R}_b}{\underline{R}_b + \underline{R}_b} = \frac{\underline{R}_b}{2} = \frac{b}{a} \underline{R}_0 \tag{10}$$

To check the validity of $\underline{R}_a$ and $\underline{R}_b$, if we set $a = b$, we then have $\underline{R}_a = 2\underline{R}_0$ and $\underline{R}_b = 2\underline{R}_0$. The result agrees with the resistance of the square mesh discussed previously. The results of these calculations are summarized in figure 12.

We can now calculate the proper resistors to represent a rectangular section of soil of any size. It is important to notice that only the ratio of $a$ to $b$ occurs in the final relationships. For example, the same resistances would be used to represent a block of soil 1 foot by 4 feet, 2 by 8 feet, or 3 by 12 inches. Each block has the same ratio of 1:4. In figure 13 the resistances are given for a number of commonly encountered examples.

## COMBINING "BUILDING BLOCKS" OF SOIL

After determining the values of resistances to represent each block of soil, now we wish to join them together. If we have a block of soil of dimensions 2 by 2 feet that is to be joined to a 2 by 4 foot block, the procedure is shown in figure 14. To join side $\overline{AB}$ to $\overline{CD}$, it is essential that $\overline{AB}$ and $\overline{CD}$ must have the same dimensions, i.e. of 2 units each. To combine $\overline{AB}$ and $\overline{CD}$ in parallel, the resistance to be used at this boundary is given by equation 11.

$$\begin{aligned} \underline{R}_{\overline{AC}\,\overline{BD}} &= \underline{R}_{AB} // \underline{R}_{CD} \\ &= 2\underline{R}_0 // \underline{R}_0 = \frac{2\underline{R}_0\underline{R}_0}{2\underline{R}_0 + \underline{R}_0} = \frac{2}{3} \underline{R}_0 \end{aligned} \tag{11}$$

The physical significance of the combined blocks shown in figure 14, c can be summarized as follows:

1. The two blocks now have an outside dimension of 2 feet by (2 + 4 feet), or 2 by 6 feet.

2. Since it now has a dimension of 2 by 6 feet, the total resistance in the horizontal direction (from $\overline{KL}$ to $\overline{MN}$) $\underline{R}_H = 3\underline{R}_0$ and the total resistance in the vertical direction (from $\overline{KM}$ to $\overline{LN}$) $\underline{R}_V = \frac{\underline{R}_0}{3}$.

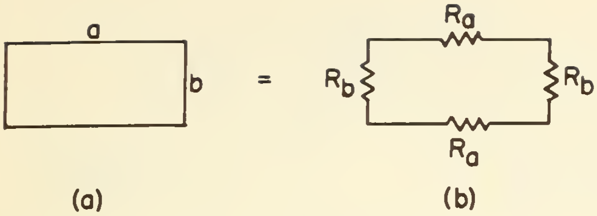



FIGURE 9.—Representing a rectangular block of soil by four resistors.

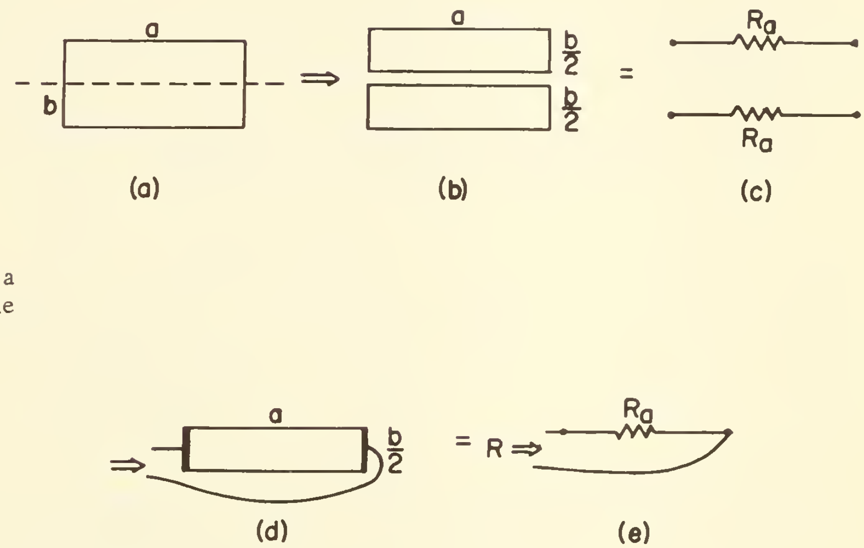



FIGURE 10.—Representation of a rectangular block of soil in the horizontal direction.

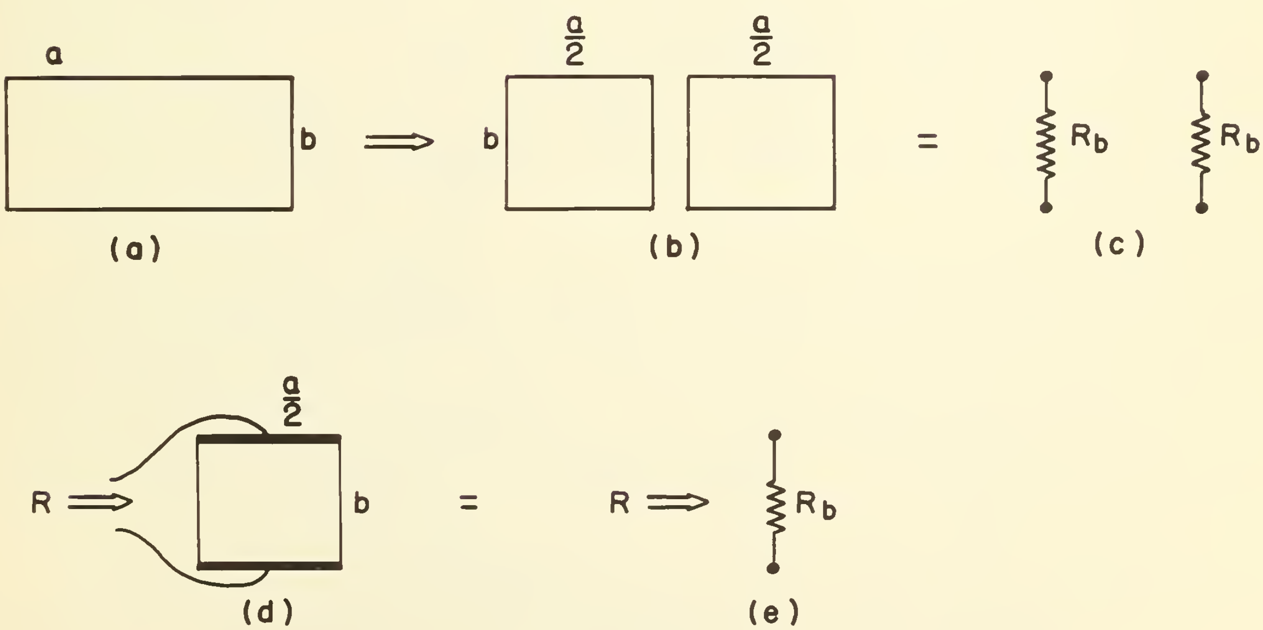



FIGURE 11.—Representation of a rectangular block of soil in the vertical direction.

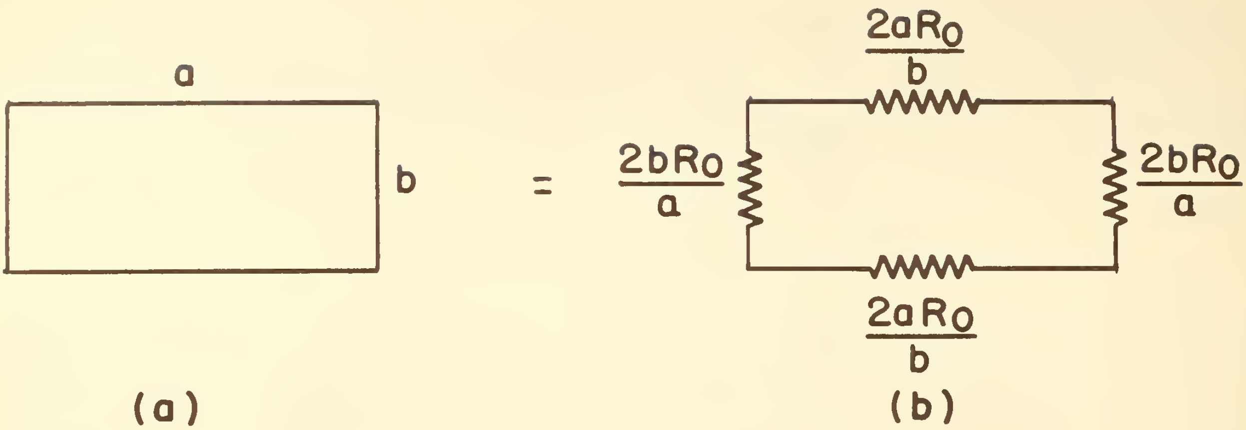



FIGURE 12.—Representation of a rectangular block of soil with dimensions a by b, using four resistors.

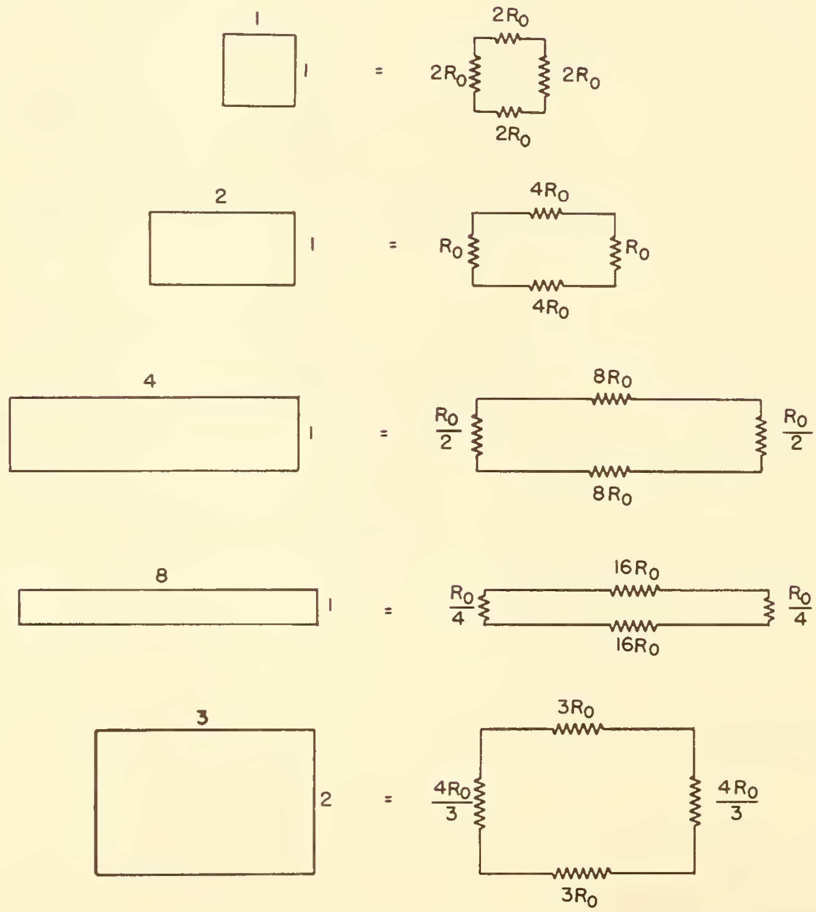



FIGURE 13.—Representation of rectangular soil blocks having different dimensions with meshes of resistors.

3. The outside resistances are unchanged. Each side portion can be joined to another block along the side of the same dimension. For example, $\overline{KAC}$ can be joined to another block along the side dimension of 2 feet and side $\underline{ACM}$ with the side dimension of 4 feet, etc.

There is another way to determine the combined resistance $\overline{AC}\ \overline{BD}$. This is done by a reverse analysis of the circuit. If we redraw figure 14, c by designating resistance $\overline{AC}\ \overline{BD}$ as $\underline{R}_1$ (fig. 15, a) and short out points $\underline{K}$, $\overline{AC}$, $\underline{M}$, and likewise points $\underline{L}$, $\overline{BD}$, $\underline{N}$, we will have three resistors joined in parallel as shown in figure 15, b. The overall resistance $\underline{R}_V$ will be equal to $\frac{\underline{R}_o}{3}$ as given by the first expression of equation 12. We can obtain the value of $\underline{R}_1$ by utilizing the remaining expressions in equation 12.

$$\underline{R}_V = \frac{\text{length}}{\text{width}} \underline{R}_o = \frac{2}{6} \underline{R}_o = \frac{\underline{R}_o}{3}$$

$$\frac{\underline{R}_o}{3} = 2\underline{R}_o /\!/ \underline{R}_1 /\!/ \underline{R}_o \tag{12}$$

$$\text{or} \quad \frac{1}{2\underline{R}_o} + \frac{1}{\underline{R}_1} + \frac{1}{\underline{R}_o} = \frac{3}{\underline{R}_o}$$

$$\underline{R}_1 = \frac{2}{3} \underline{R}_o$$

To check the value of $\underline{R}_H$, we short out points $\underline{KL}$ and $\underline{MN}$. We will then have the circuit as shown in figure 16, a. Notice that this circuit forms a Wheatstone bridge so that the potential at points $\underline{AC}$ and $\underline{BD}$ are equal and we can disregard the resistor $\overline{AC}\ \overline{BD}$. We now have the circuit as shown in figure 16, b and which is analyzed by the expression in equation 13.

$$\underline{R}_H = (2\underline{R}_o + 4\underline{R}_o) /\!/ (2\underline{R}_o + 4\underline{R}_o)$$

$$\therefore \frac{1}{\underline{R}_H} = \frac{1}{6\underline{R}_o} + \frac{1}{6\underline{R}_o} \tag{13}$$

$$\underline{R}_H = 3\underline{R}_o$$

The reverse analysis of a circuit is very helpful when we have to join blocks of soil along sides that are different in dimension. The situation arises, for example, when we wish to expand the fine meshes of the resistance network near the drain region to the coarse meshes in the region farther away from the drain. This expansion is done for both practical and economical reasons, as it has been found that in a drainage system most of the potential loss occurs near the drain. Since more detailed information is needed near the drain, a finer mesh is used in this region than in the region farther away from the drain.

There is a certain type of building block that will permit the expansion of a fine mesh to a coarse one. The building blocks for this special boundary are not so easy to simulate as one may first think. As shown in figures 17, a and 17, b, the network is to be expanded from small blocks of soil ($\underline{D}$ and $\underline{E}$) with

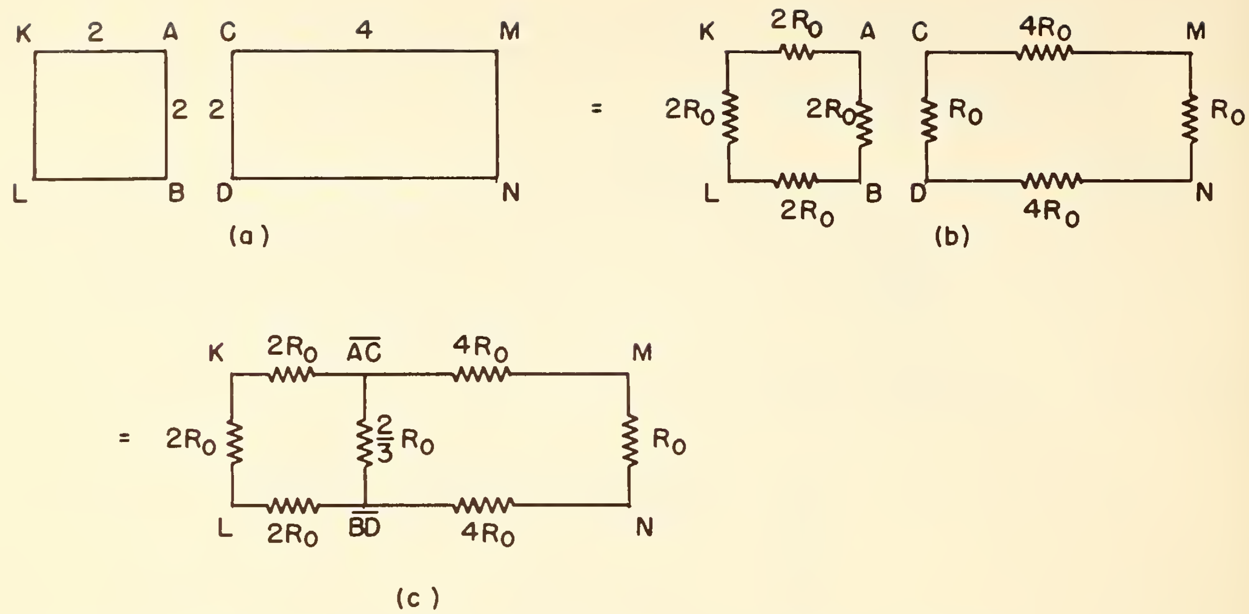



FIGURE 14.—Joining two meshes of resistances that represent two rectangular blocks of soil.

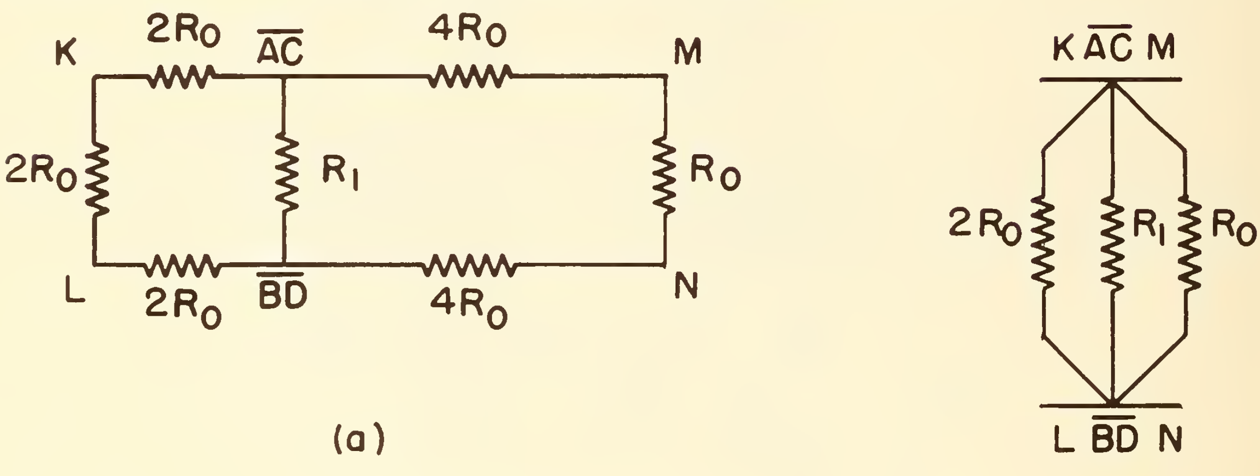



FIGURE 15.—"Reverse analysis" to determine the resistance $\overline{AC}\ \overline{BD}$. (Also see fig. 14.)

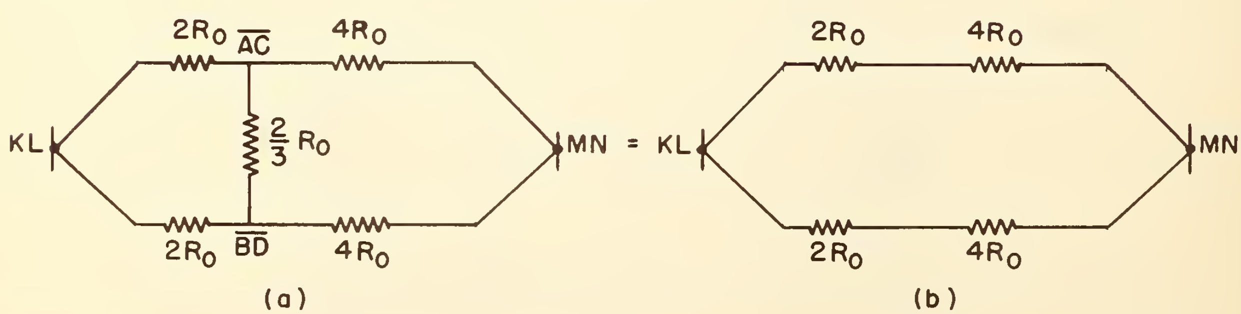



FIGURE 16.—Reverse analysis to confirm the horizontal resistance $R_H$ of the circuit shown in figure 15, a.

dimensions 1 by 1 to larger blocks (F) with dimensions 2 by 6. In such a case, we introduce blocks, A, B, and C into the picture to function as boundary building blocks between a fine mesh D or E and a coarse mesh F. In actuality, blocks A, B, and C must be combined to a single block with the right side to be joined to the coarse mesh and the left side to be joined to the fine mesh. The problem arises as how to join a side dimension of block A and B to the side dimension of block C.

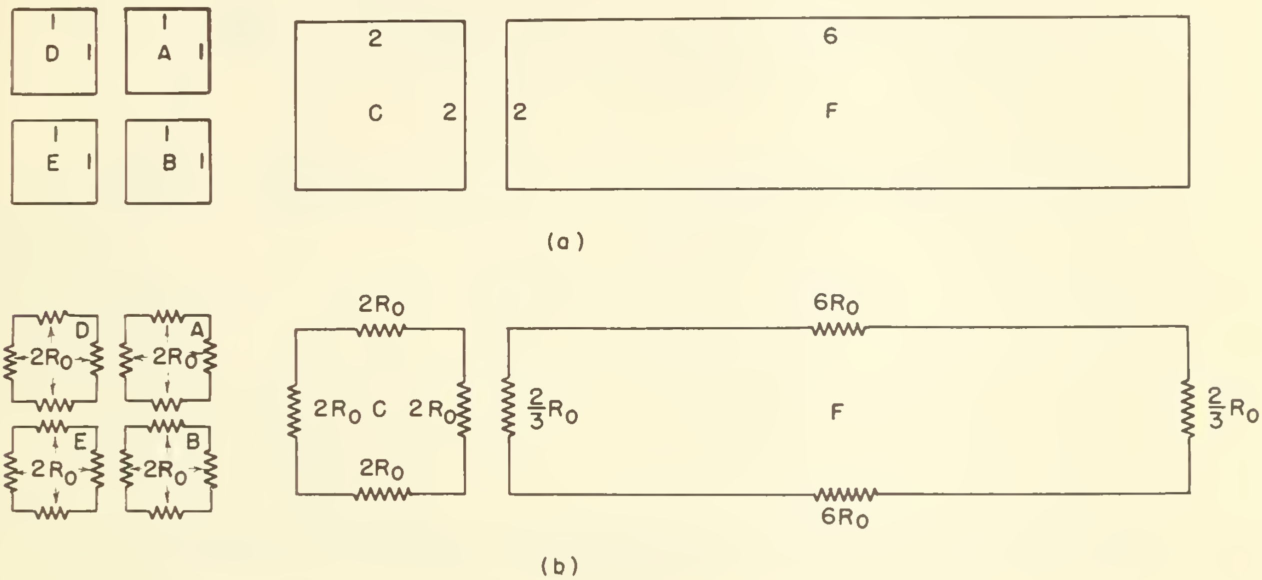



FIGURE 17.—Expansion from region of fine meshes to one of coarse meshes.

To solve such a problem, we have to attack it indirectly. We cannot say that we "join" them together in the sense that blocks were joined in the preceding section. Rather, we have to look at the "finished product" and analyze it in reverse. For example, the finished product of the boundary building block is shown in figure 18, a, which can be represented by a network of resistors as shown in figure 18, b. All outside resistors are unaltered from the original value of $2R_o$.

As shown in figures 18, c and d, the finished block has an outside dimension of 3 by 2 feet. From equations 8 and 10 the total resistance in the horizontal direction is $R_H = (3/2)R_o$ and that in the vertical direction is $R_V = (2/3)R_o$. With the outside resistances in figure 18, b, being equal to $2R_o$, they are ready to be joined to the fine mesh on the left with side dimension of 1 and to the coarse mesh on the right with side dimension of 2 feet. Now we must determine $R_1$, $R_2$, and $R_3$ in figure 18, b so that the overall resistance in the horizontal direction $R_H$ is equal to $(3/2)R_o$ and the overall resistance in the vertical direction $R_V$ is equal to $(2/3)R_o$, which are the original values of the resistances representing a 2- by 3-foot block of soil.

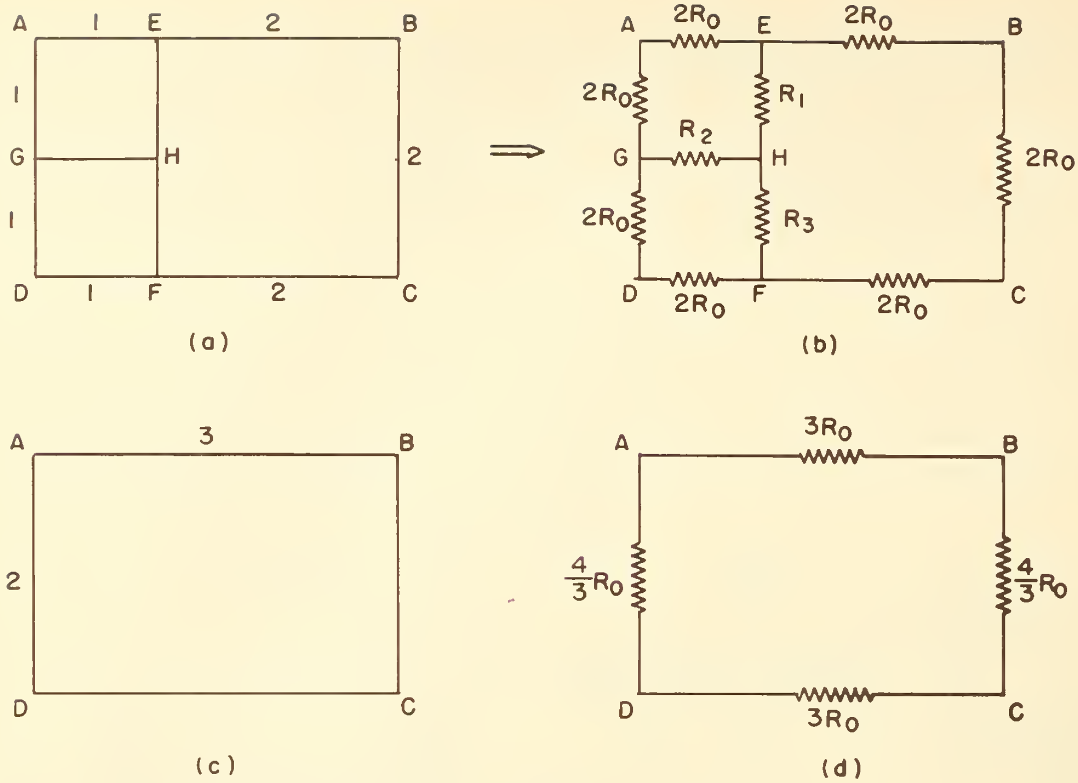



FIGURE 18.—Representing boundary building blocks between fine and coarse meshes in the resistance network.

In determining $R_1$, $R_2$, and $R_3$, we have only two equations, i.e., the equations for $R_H$ and $R_V$. However, it is possible to set $R_1 = R_3$ because of symmetry. We now have the network as shown in figure 19, a. To formulate the equation for $R_V$, we short out points AEB and DFC. We will then have the network as shown in figure 19, b. Notice also that the circuit G, AEB, H, DFC forms a Wheatstone bridge so that the potential at points G and H are equal. Therefore, we can disregard the resistance $R_2$ in the calculation. We now have the circuit as shown in figures 19, c and the magnitude of $R_1$ is obtained by the following:

$$(2R_0 + 2R_0) /\!/ (R_1 + R_1) /\!/ 2R_0 = R_V = \frac{2}{3} R_0$$

$$\therefore \frac{1}{R_V} = \frac{1}{4R_0} + \frac{1}{2R_1} + \frac{1}{2R_0} = \frac{3}{2R_0} \tag{14}$$

$$\text{or} \quad R_1 = \frac{2}{3} R_0$$

To solve for $R_2$, short out points AGD and BC of figure 19, a. The new network is shown in figure 20, a. This network can also be thought of as two sets of equivalent resistors AEB and AFB, connected in parallel (See fig. 20,

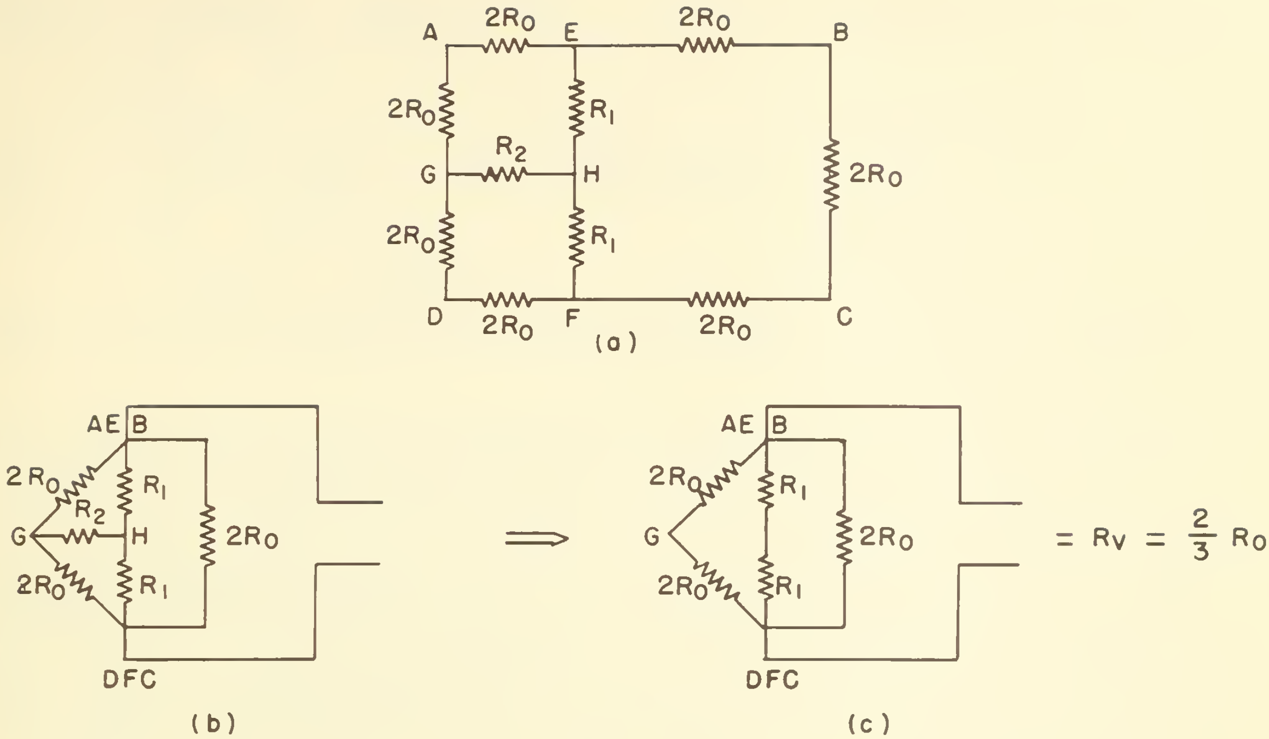



FIGURE 19.--Circuit analysis to determine the Resistance $\underline{R}_1$ at the boundary between fine and coarse meshes in a network.

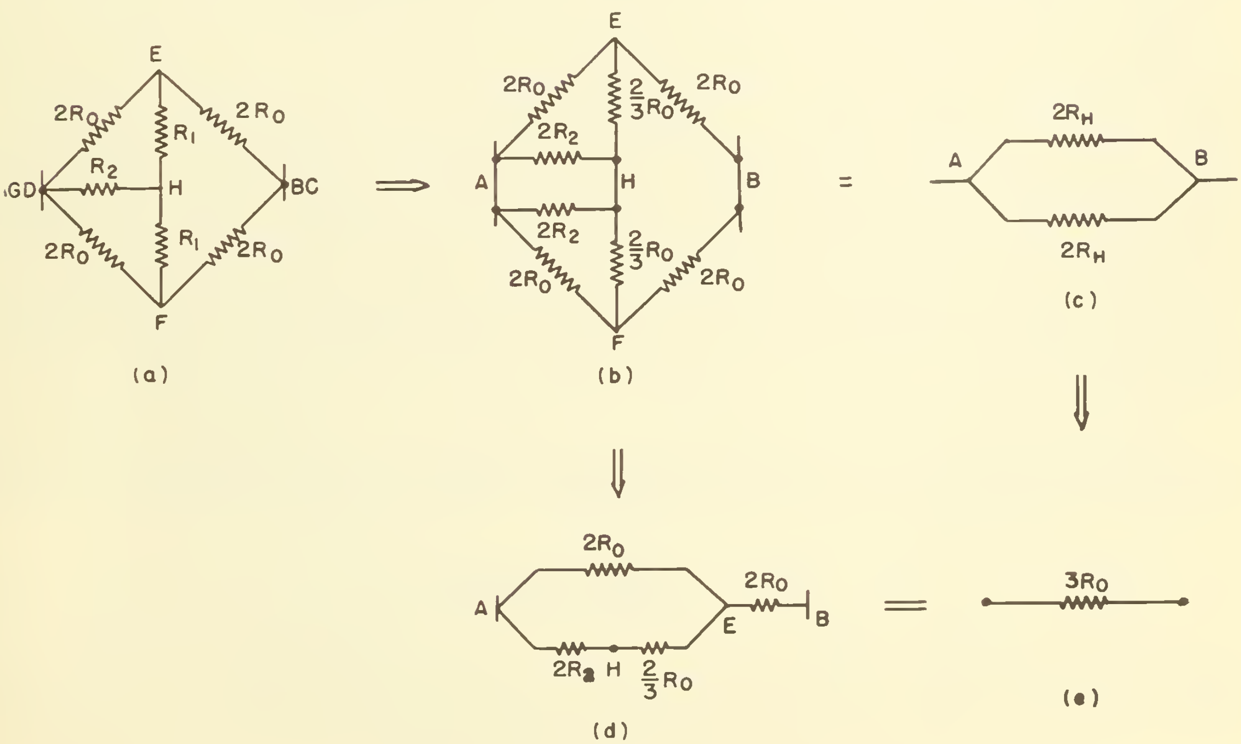



FIGURE 20.--Circuit analysis to determine the resistance $\underline{R}_2$ as shown in figure 18, b and in figure 19, a.

b and c), so that each set will have a resistance of $2\underline{R}_H$. Therefore, set $\underline{AEB}$ can be analyzed by equation 15 (fig. 20, d and 20, e):

$$[(2\underline{R}_2 + \frac{2}{3}\underline{R}_O)//2\underline{R}_O] + 2\underline{R}_O = 2\underline{R}_H = 3\underline{R}_O \qquad (15)$$

$$\therefore \underline{R}_2 = \frac{2}{3}\underline{R}_O$$

Notice that until now nothing is said about the inside dimensions of $\underline{GH}$, $\underline{EH}$, and $\underline{HF}$ of figure 18, a.

An interesting point is observed with regard to node $\underline{H}$ in the interior of this building block (fig. 21, a). The question arises as to where to plot the value of the voltage measured at this internal node. One's first inclination might be to plot it at the intersection designated by $\underline{H}$ in figure 21, b. It can be shown, however, that the actual point of correspondence is located at $\underline{X}$. Consider the situation shown by figure 21, c. Let us short out $\underline{AGD}$ and $\underline{BC}$ and then apply 3 volts at $\underline{BC}$ while $\underline{AGD}$ is attached to the ground (i.e., at zero volt). If we measure the potential at $\underline{E}$ or $\underline{F}$, we will have a reading of 1 volt, indicating that both $\underline{E}$ and $\underline{F}$ are located one third of the distance from $\underline{AD}$ to $\underline{BC}$ as expected. If we then measure the voltage at $\underline{H}$, we will have a reading of 2/3 volt, indicating that $\underline{GH}$ (or in reality $\underline{GX}$ of fig. 21, b) is 2/3 the distance of either $\underline{AE}$ or $\underline{DF}$. The potential of 2/3 volt at point $\underline{H}$ can also be calculated, if we use Kirchhoff's law having the voltages at adjacent nodes $\underline{E}$, $\underline{F}$, and $\underline{G}$ as 1, 1, and 0 volt, respectively, with the resistance $\underline{EH}$, $\underline{FH}$, and $\underline{GH}$ each being $(2/3)\underline{R}_O$. Therefore, in the analog system, the voltage measured at the node $\underline{H}$ in figure 21, a should be plotted at point $\underline{X}$ located 2/3 the distance between $\underline{GH}$, as shown in figure 21, b. The finished product of the 2- by 3-foot block of soil is shown in figure 21, d, and its network of resistors is shown in figure 21, a. This same circuit was described by Liebmann (6, pp. 97, figs. 10 and and 11). If one analyzes Liebmann's circuit, one will find that $\underline{R}_V = (2/3)\underline{R}_O$. However, $\underline{R}_H$ of Liebmann's circuit will be equal to $(5/3)\underline{R}_O$ instead of $(3/2)\underline{R}_O$, which represents a discrepancy of 11.1 percent. Also, if a potential of 3 volts is applied across $\underline{BC}$ and $\underline{AGD}$ of Liebmann's circuit, one would find that the voltage at point $\underline{E}$ or $\underline{F}$ is 1.2 volts and the voltage at point $\underline{H}$ is 0.6 volt. These values indicate that the distance $\underline{AE}$ or $\underline{DF}$ is 2/5 of the distance $\underline{AB}$ instead of the value of 1/3.

The value of the resistance $\underline{GH}$ of figure 21, a, can be used to locate point $\underline{X}$ of figure 21, d. In all cases the resistance $\underline{R}_{AE}$ is equal to $\underline{R}_{DF}$. The distance $\underline{GX}$ is related to distance $\underline{AE}$ as shown in equation 16.

$$\underline{GX} = (2\underline{R}_{GH}/\underline{R}_{AE})\,\underline{AE} \qquad (16)$$

Further illustrations will be shown later on in the discussion of anisotropic soil.

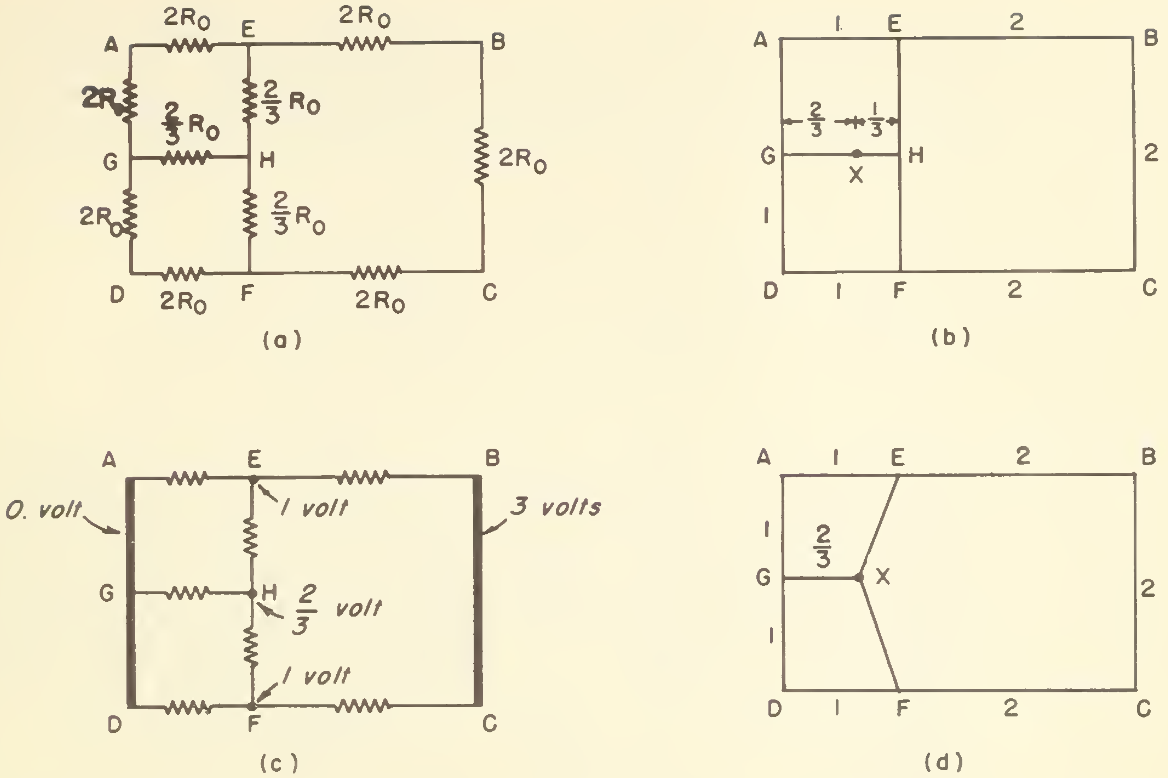



FIGURE 21.—Locating the interior point X in the soil block that corresponds to the point H in the network.

With the information on different types of building blocks discussed above, we are now ready to put them together to form a network representing a homogeneous soil medium. For example let us represent a section of soil having dimensions of 5 by 20 by 1. The dimensional units are arbitrary, but we will consider the soil to have dimensions of 5 by 20 feet by 1 foot. The soil may be represented by the building blocks of soil shown in figure 22, a with side dimensions as indicated. Figure 22, b gives the resistances representing the building blocks, while figure 22, c shows the assembled network. The arrangement of resistance shown in figure 22, c will yield the most accurate information on potentials along the left-hand boundary where the meshes are smallest. This conclusion follows, as potentials in a network can only be measured at the nodes. This arrangement is particularly useful when a subsurface drain is located in the region of small meshes. Since the potential drop is quite large near a sink (i.e., the drain), greater accuracy is needed in that region.

In a later section (p. 24) the simulation of building blocks in the region near a circular drain will be discussed. It will be shown that those resistances which represent the drain can be inserted at any depth on the left side of the network shown in figure 22, c. If a half-drain were placed on the left boundary and constant (but different) potentials applied at the drain and at the top boundary of the network, one would simulate ponded flow into a drain. The left-hand side of the network would represent the vertical plane through the center of the

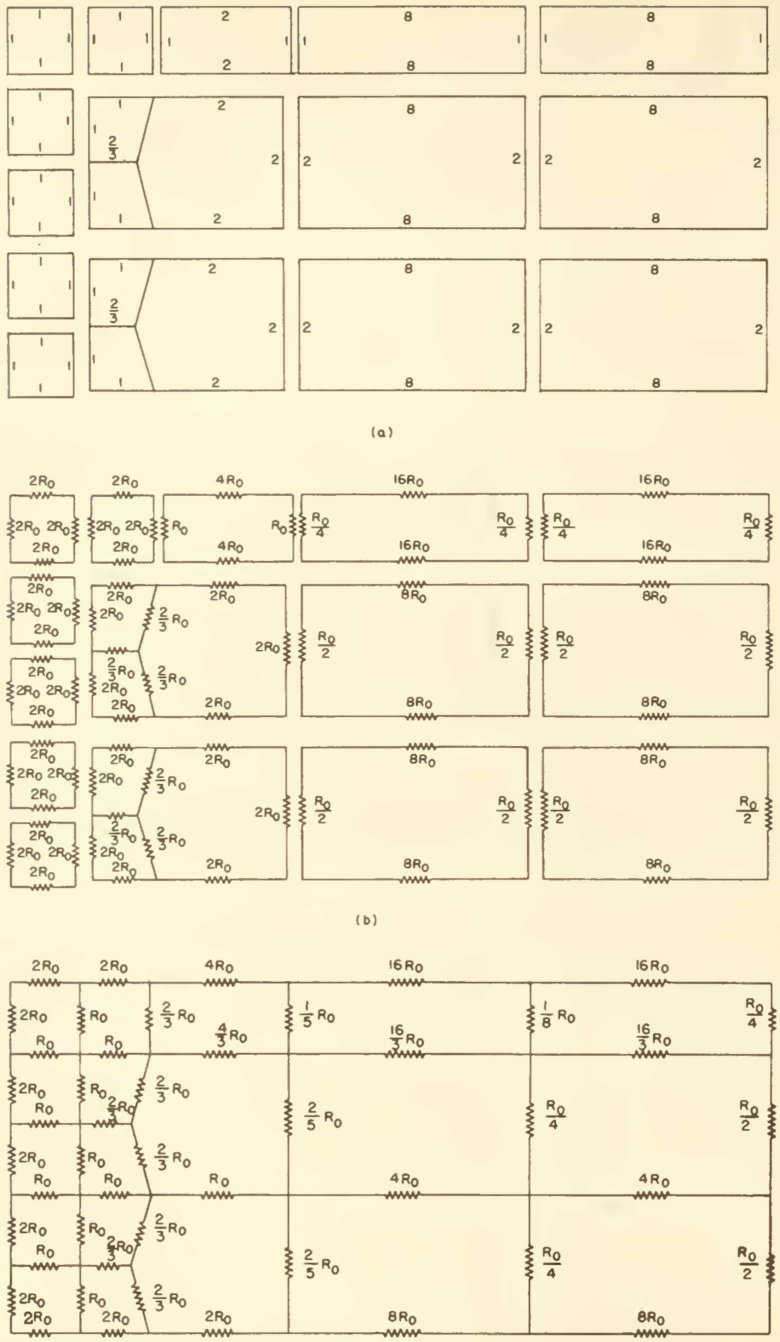



FIGURE 22.—Representing homogeneous soil sections with a resistance network.

drain. The top part of the network would represent the soil surface, and the bottom part, an impervious layer. The right boundary of the network would represent a vertical plane across which flow does not occur and would correspond to the plane between equally spaced drains in level topography (9).

Observe that in figure 22, a, if we use a mesh size of 1 foot, we will have the impervious layer at 5 feet. If we wish to locate the impervious layer at 4 feet, we can remove the top row of building blocks. With this type of arrangement, we can build the network with odd or even depth to the impervious layer by adding or removing the top row accordingly.

## REPRESENTING A STRATIFIED SOIL

So far, all the concepts deal with homogeneous medium where the entire soil is of the same hydraulic conductivity $\underline{K}$. In actuality, the soil profile is divided into different horizons. Generally, each horizon has a different conductivity. For simplicity, we will choose a soil profile having three layers of different conductivities: $\underline{K}_1$, $\underline{K}_2$, and $\underline{K}_3$, respectively, from the top layer to the bottom one. Also, we will designate $\underline{K}_1$, $\underline{K}_2$, and $\underline{K}_3$ as 2.24, 0.56, and 0.28 cm./hr. The ratios $\underline{K}_1 : \underline{K}_2 : \underline{K}_3$ will be 8:2:1. These particular values are chosen so that $\underline{K}_1/\underline{K}_2 = 4$ and $\underline{K}_2/\underline{K}_3 = 2$. Let the characteristic resistance $\underline{R}_o$

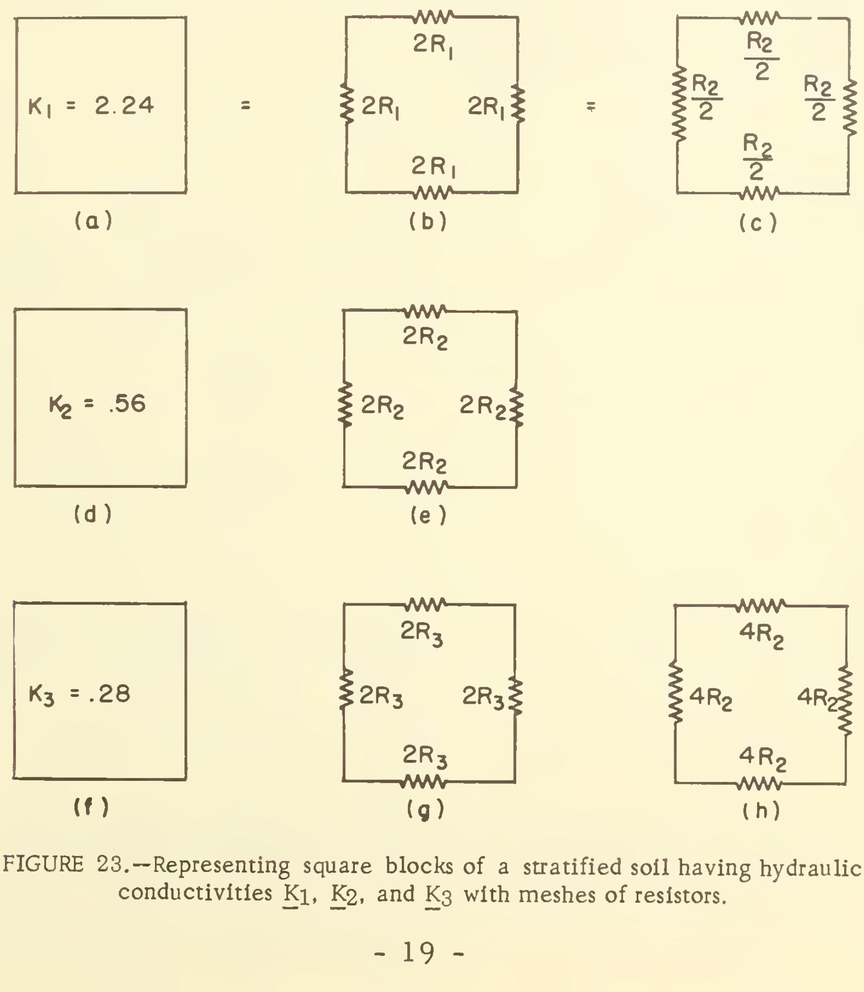



FIGURE 23.--Representing square blocks of a stratified soil having hydraulic conductivities $\underline{K}_1$, $\underline{K}_2$, and $\underline{K}_3$ with meshes of resistors.

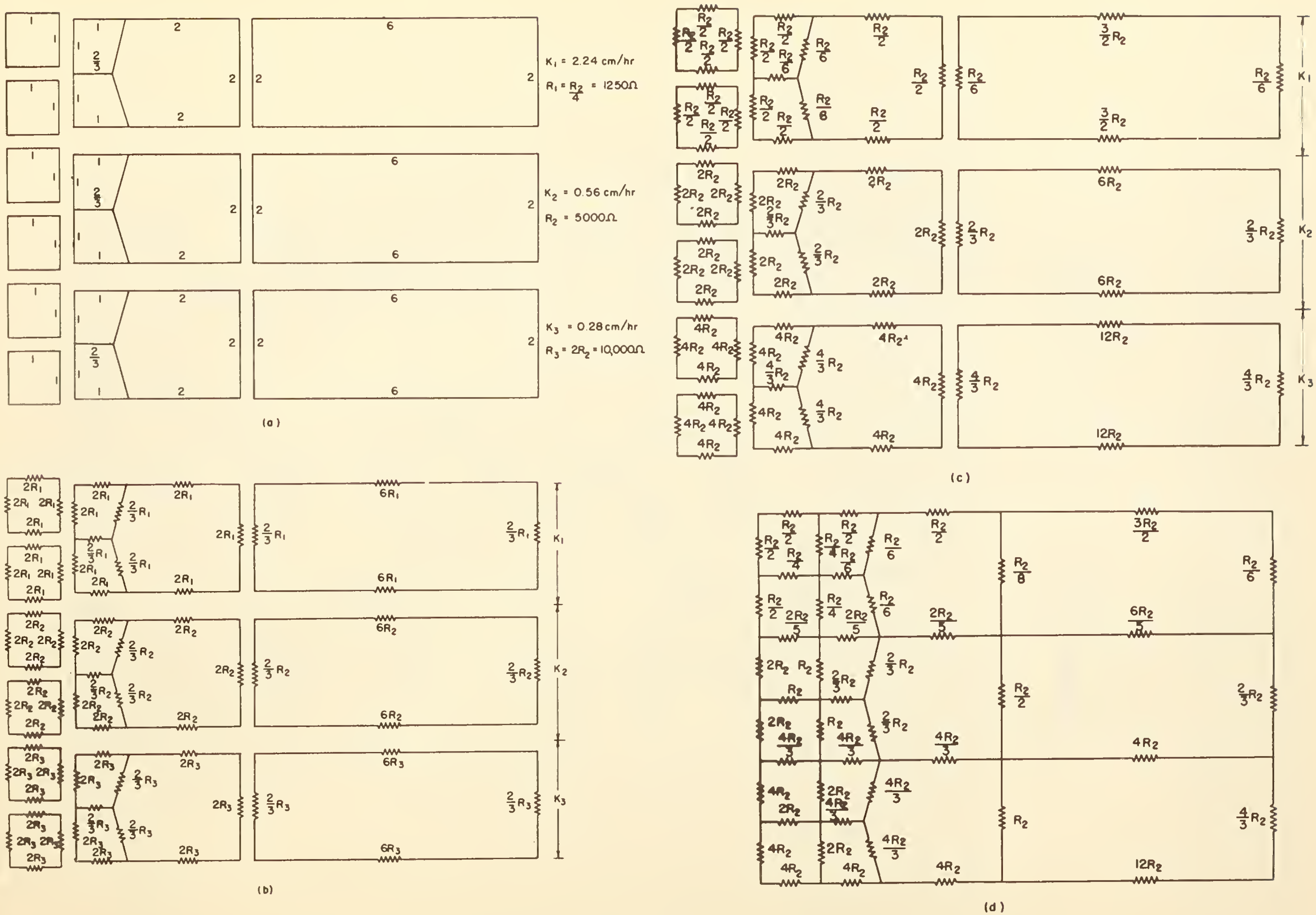



FIGURE 24.—Representing sections of a 3-layered stratified soil with a resistance network.

be 5,000 ohms and equal to $\underline{R}_2$ in order to represent $\underline{K}_2$. We can see that $\underline{R}_1$ (representing $\underline{K}_1$) is 5,000/4 or 1,250 ohms and $\underline{R}_3$ (representing $\underline{K}_3$) is 5,000 x 2 or 10,000 ohms. In the $\underline{K}_1$ region, a square block of soil can be represented by four resistors as shown in figure 23, b. Figures 23, e and g, represent a square block of soil in the $\underline{K}_2$ and $\underline{K}_3$ regions, respectively. Since 5,000 ohms is used as characteristic resistance of a square mesh, it is more convenient to express both $\underline{R}_1$ and $\underline{R}_3$ in terms of $\underline{R}_2$. Figures 23, c and h, show the transformations. An example of the networks for stratified layers is shown in figure 24.

A more general expression for calculating $\underline{R}_H$ and $\underline{R}_V$ is given in the appendix, part II, for the case where rectangular meshes of different sizes are employed and the hydraulic conductivity also differs in the various horizontal layers.

## REPRESENTING AN ANISOTROPIC SOIL

In the previous discussion we were dealing with soil that had a different hydraulic conductivity in each layer or horizon. It is sometimes found that

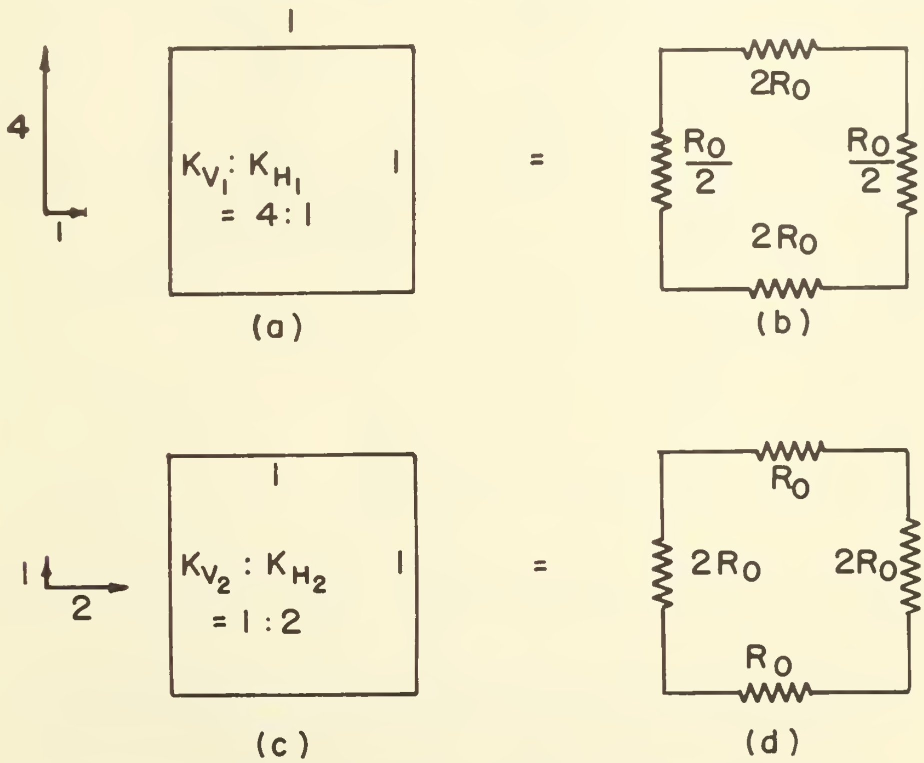



FIGURE 25.—Representing square blocks of soil having different hydraulic conductivities in the vertical and horizontal directions (anisotropic soil).

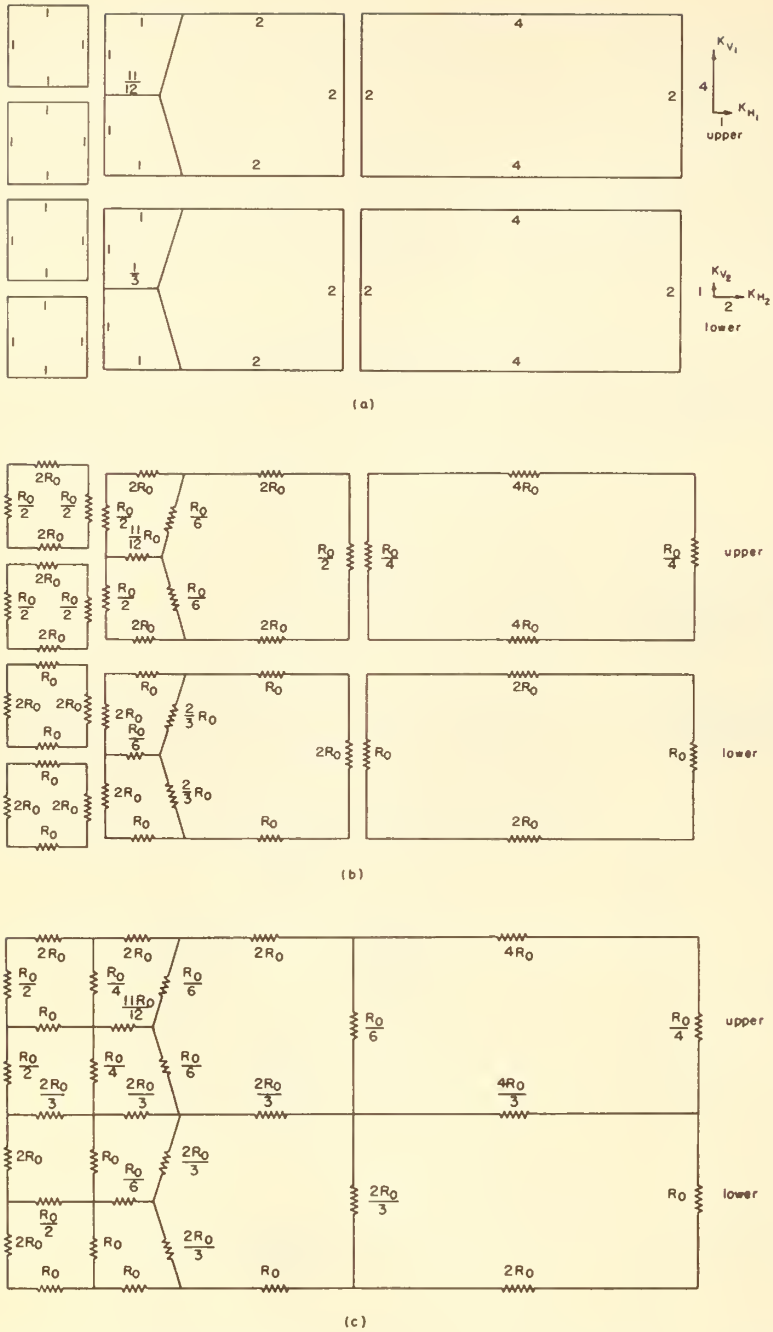



FIGURE 26.—Representing sections of anisotropic soil having two layers with a resistance network.

within the same layer of soil, the hydraulic conductivities in the horizontal and in the vertical directions are also different. This type of soil is called anisotropic with respect to its hydraulic conductivity. To set up a network to represent an anisotropic soil, let us designate a soil with two layers. The upper layer has a hydraulic conductivity in the vertical direction, $K_{V1}$ as 0.12 cm./hr., and in the horizontal direction, $K_{H1}$ as 0.03 cm./hr. In the lower layer, $K_{H2}$ = 0.06 cm./hr. and $K_{V2}$ = 0.03 cm./hr. By designating the conductivity of 0.03 cm./hr. as 1, we can see that the ratio $K_V$:$K_H$ of the upper layer is 4:1 and of the lower layer is 1:2 (fig. 25). By choosing the characteristic resistance of

5,000 ohms ($R_o$) to represent the conductivity of $K_{H1}$ and $K_{V2}$ (each equaling 0.03 cm./hr.), we can see that resistance $R_{V1}$ representing $K_{V1}$ is 1,250 ohms, whereas the resistance $R_{H2}$ representing $K_{H2}$ is 2,500 ohms. In the upper layer, a square block of soil can be represented by four resistors as shown in figures 25, a and b, and in the lower layer by figures 25, c and d. An assembled network of resistors for this soil is shown in figures 26, a, b, and c. The over-all distances of the simulated flow medium are arbitrarily selected.

Special attention is called to the location of the internal nodes of the boundary building block between the coarse mesh. As shown in figure 26, a, the internal node of the boundary building block of the upper layer is plotted at the point 11/12 of 1 and that of the lower layer at 1/3 of 1. These locations must be determined by either calculation or experimentation as shown in a previous section, p. 16.

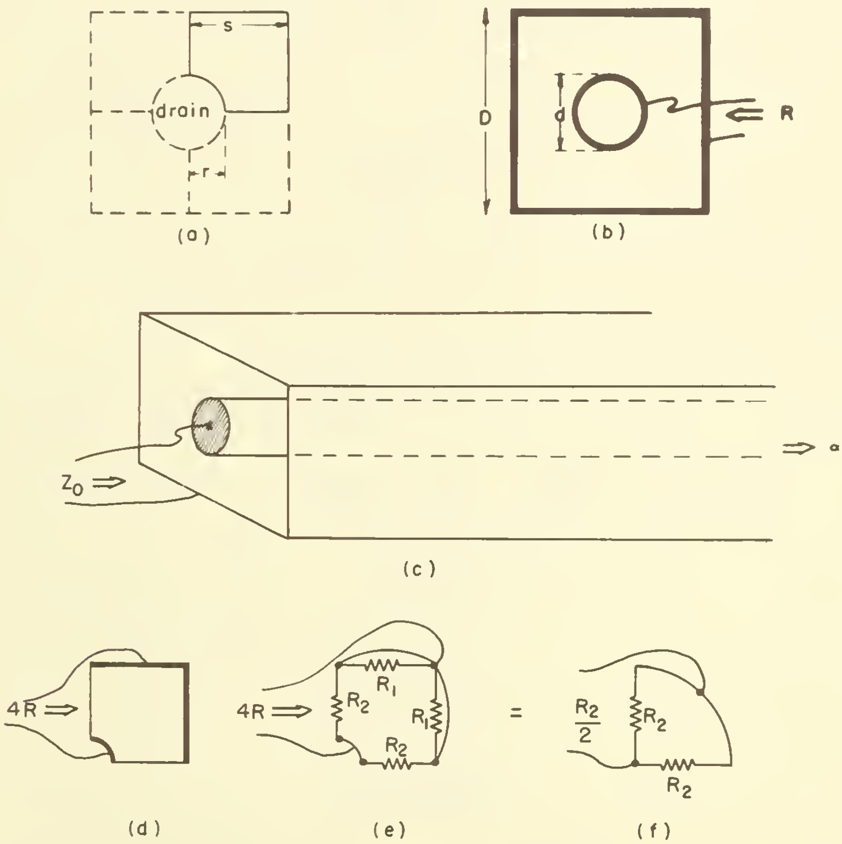



FIGURE 27.—Diagram showing the approach used to calculate the resistance $R_2$ see (e) which represents the soil region around a drain. The value $R_2$ is equal to $8R$, where $R$ is the resistance of the conducting paper shown in (b). The value of $R$ is evaluated by using the analogy between $R$ and the impedance $Z_o$ of the transmission line shown in (c). The relationship between $R$ and $Z_o$ is expressed by equations 17 and 18.

## REPRESENTING A CIRCULAR DRAIN

Up to this point building blocks always represent rectangular sections of soil. In studies dealing with subsurface drains, however, the drain is usually circular in cross section. To cover this case, we must extend the building blocks concept to include such curved sections. To approach this problem, first cut from a piece of resistive paper the section drawn with a solid line as shown in figure 27, a. In this figure, $\underline{r}$ is the drain radius and $\underline{s}$ is the square mesh size around the drain. At the present time it will be convenient to remember that this section is just one fourth of the entire section as shown in figure 27, a. If we paint the boundaries of the square and the circle with conducting paint as shown in figure 27, b, we could then measure the resistance $\underline{R}$. However, direct measurement is sometimes neither convenient nor accurate. This problem has been solved mathematically for electrical transmission lines and wave guides. The resistance $\underline{R}$ of figure 27, b, is essentially the same as the impedance $\underline{Z}_o$ of a square wave guide with a circular inner conductor as shown in figure 27, c. The only difference between $\underline{R}$ and $\underline{Z}_o$ is a conversion factor $\underline{Z}'_o$ relating the characteristic resistance of free space and the characteristic resistance (resistance per square = $\underline{R}_o$) of the resistive paper. The relationship is given by equation 17 (5, pp. 426-429).

$$\underline{R} = \frac{R_o}{\underline{Z}'_o} \underline{Z}_o \tag{17}$$

The expression yielding $\underline{Z}_o$ for a transmission line as shown in figure 27, c is given by equation 18 (12, p. 590).

$$\begin{aligned}
\underline{Z}_o &\approx 138 \log_{10}\rho + 6.48 - 2.34\underline{A} - 0.48\underline{B} - 0.12\underline{C} \\
\text{where } \rho &= \frac{D}{d} \quad \text{(Fig. 27b)} \\
\underline{A} &= \frac{1 + 0.405\rho^{-4}}{1 - 0.405\rho^{-4}} \\
\underline{B} &= \frac{1 + 0.163\rho^{-8}}{1 - 0.163\rho^{-8}} \\
\underline{C} &= \frac{1 + 0.067\rho^{-12}}{1 - 0.067\rho^{-12}} \\
\text{and } \underline{Z}'_o &= \sqrt{\frac{\mu_o}{\epsilon_o}} \approx 120\pi = 376.7 \text{ ohms where} \\
\mu_o &= \text{permeability of free space} \\
\epsilon_o &= \text{permitivity of free space}
\end{aligned} \tag{18}$$

In our network, if we are dealing with a square mesh around the drain with mesh size $\underline{s}$ and radius of the drain $\underline{r}$, we see that $\underline{s} = \frac{D}{2}$ and $\underline{r} = \frac{d}{2}$. Therefore $\rho = \frac{D}{d} = \frac{s}{r}$. From these expressions, we can determine $\underline{R}$ of figure 27, b.

Returning to the original problem, if we paint conducting lines on the outside and inside boundaries of figure 27, d and then measure the resistance, we will get $4\underline{R}$ ohms since this part represents only one of the four quarters that are in parallel, as shown in figure 27, a. The resistance of the section shown in figure 27, d can be represented by the resistance shown in figure 27, e. In this particular measurement, the resistors of value $\underline{R_1}$ are effectively shorted out so that the circuit reduces to figure 27, f. Taking an overall view of figure 27, d leads us to the relationship:

$$4\underline{R} = \frac{\underline{R_2}}{2}, \text{ or}$$
$$\underline{R_2} = 8\underline{R} = \frac{8\underline{Z_0}}{\underline{Z_0'}}\underline{R_0} \tag{19}$$

The next question to ask is what value to pick for the $\underline{R_1}$ resistors in figure 27, e? In our studies, the value of $\underline{R_1} = 2\underline{R_0}$ is chosen. In every way this value seems reasonable, but there is no basis to assume that it is exact. It may very well depend on the diameter of the drain as does $\underline{R_2}$. However, a number of experiments using $\underline{R_1}$ equal to $2\underline{R_0}$ have yielded results (11) which are in good agreement with exact analytical solutions (3, equation 11).

To illustrate the above approach in calculating the resistances to represent the drain region, let us designate the size of the mesh as 1 foot, radius of the drain $\underline{r}$ = 3 inches, $\underline{R_0}$ = 5,000 ohms.

$$\underline{Z_0} = 138 \log_{10}\rho + 6.48 - 2.34\underline{A} - 0.48\underline{B} - 0.12\underline{C}$$

$$\text{where } \rho = \frac{12}{3} = 4$$

$$\underline{A} = 1.00317$$

$$\underline{B} \approx 1$$

$$\underline{C} \approx 1 \tag{19a}$$

$$\underline{Z_0} = 138 \log_{10}4 + 6.48 - 2.3474 - 0.48 - 0.12 = 86.6169$$

$$\underline{R_2} = \frac{8\underline{Z_0}}{\underline{Z_0'}}\underline{R_0} = \frac{8 \times 86.6169}{376.7} \times 5000 = 9197 \text{ ohms}$$

$$\frac{\underline{R_2}}{2} = 4599 \text{ ohms}$$

From the above example, we can see that for a given value of $\rho$ we can calculate the drain resistance $R_2 = \frac{8Z_o}{Z'_o} R_o$. If we define the constant $C_d = \frac{8Z_o}{Z'_o}$, the drain resistance is given by $R_2 = C_d R_o$. Table 1 reports a few values of $C_d$ along with the corresponding values of $\rho$. The values reported therein will suffice for calculating those values of the drain resistance that are normally encountered in network studies.

Table 1.--Some values of the constant $C_d$ as a function of $\rho$. The function $\rho$ is equal to the ratio $s/r$, where $s$ is the size of the square mesh at the drain and $r$ is the drain radius. (See fig. 27, a.) The magnitude of the drain resistance $R_2$, as shown in figure 28 is given by $R_2 = C_d R_o$

| $\rho = \frac{s}{r}$ | $C_d = \frac{8Z_o}{Z'_o}$ | $\rho = \frac{s}{r}$ | $C_d = \frac{8Z_o}{Z'_o}$ |
|---|---|---|---|
| 1.0000 | 0.00319 | 2.1818 | 1.06632 |
| 1.0435 | 0.07491 | 2.4000 | 1.18823 |
| 1.0909 | 0.14598 | 2.6667 | 1.32278 |
| 1.1429 | 0.21300 | 3.0000 | 1.47298 |
| 1.2000 | 0.28227 | 3.4286 | 1.64316 |
| 1.2632 | 0.35321 | 4.0000 | 1.83949 |
| 1.3333 | 0.42635 | 4.8000 | 2.07162 |
| 1.4118 | 0.50261 | 6.0000 | 2.35568 |
| 1.5000 | 0.58247 | 8.0000 | 2.72187 |
| 1.6000 | 0.66677 | 9.0000 | 2.88452 |
| 1.7143 | 0.75628 | 12.0000 | 3.23794 |
| 1.8462 | 0.85195 | 24.0000 | 4.12018 |
| 2.0000 | 0.95483 | $\infty$ | $\infty$ |

We now have the building blocks necessary to represent the area around the drain. In figure 28 we put these building blocks together to form a network in a homogeneous soil with a square mesh around the drain. (See also Appendix, pt. II.)

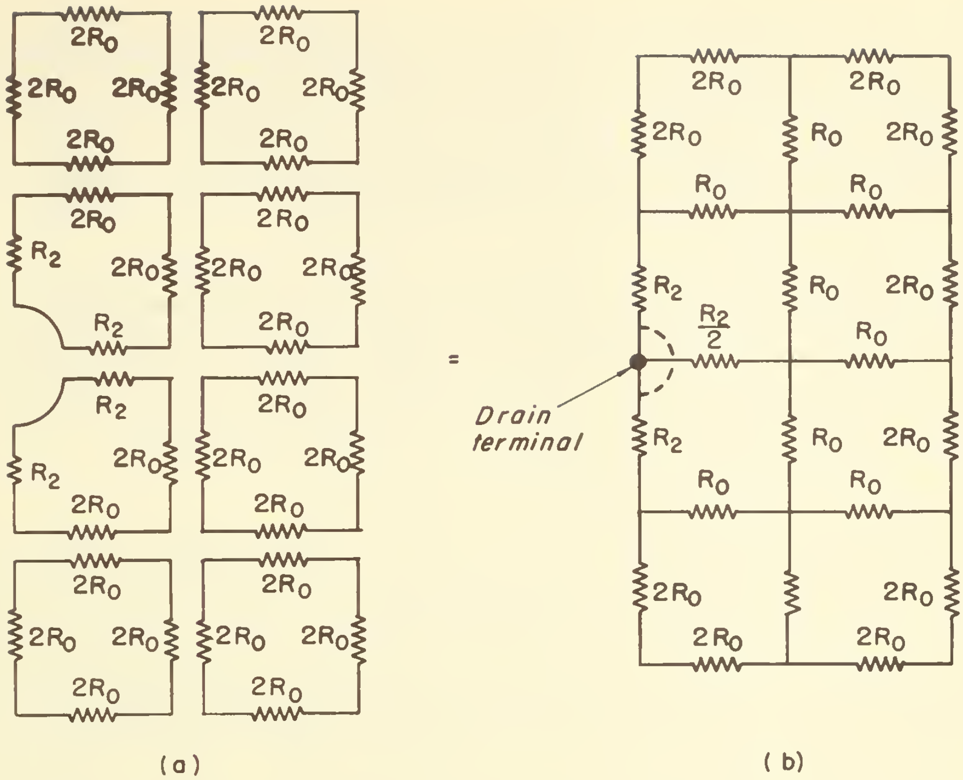



FIGURE 28.—Diagram showing connection of resistance $R_2$ which represents the drain region to those representing square soil blocks.

## AN EXAMPLE UTILIZING THE RESISTANCE NETWORK TO STUDY FLOW PROBLEMS

In network studies, there are three things in which we are primarily interested: (1) the potential distribution from which we can obtain -- among other things -- equipotential lines; (2) the streamlines from which we can learn the direction of water movement through the soil; and (3) the flow rate or, in the case of drainage, the amount of water that we can remove from the soil during a given period of time. All these data can be obtained directly with the network.

As an example, consider the ponded flow case in drainage: Multiple drains of radius r are equally spaced in a homogeneous soil at a distance a and at a depth d. Water is ponded on the level ground surface to a height t. The drains are completely permeable to water and are running full with no back pressure. An impermeable horizontal layer lies at a depth h below the ground surface. First, the network is assembled as shown in figure 29. To obtain the potential distribution, an e.m.f. (electromotive force) of 100 volts is then applied to the top boundary. The drain terminal (fig. 28) is connected to the ground as shown by the diagram in figure 29. A voltmeter is used to measure the potential at

each node. A vacuum tube voltmeter (VTVM) is recommended since its internal resistance is very high (around 11 megohms) and the potential measurement can be made accurately. For convenience, the potential difference of 100 volts is applied to the upper network boundary so that the voltage reading at any node yields the potential directly as a percentage of that at the ground surface. After the potentials are measured at each node, the equipotential lines can be drawn by interpolating between potentials at adjacent nodes.

To obtain the flowline, the boundaries of the network are now "reversed" as shown in figure 30. This is done by first removing the attachments from the top boundary and the drain terminal. This time an e.m.f. is applied to the previously unconnected boundaries as shown in figure 30. The potentials are again measured at each node, and a new set of equipotential lines are drawn. The new set of equipotential lines will be the flow lines; that is, the two sets of equipotential lines will be orthogonal.

To measure the drain flow rate, two different procedures can be followed: From equation 44 in appendix, pt. I, we see that the applied voltage $\underline{V}$ and the hydraulic head potential $\emptyset$ are related by a conversion coefficient $\underline{C}_o$. The potential at the top boundary $\emptyset_{\underline{n}}$ is equal to $\underline{d} + \underline{t}$, where $\underline{d}$ is the vertical distance between the drain center and the soil surface and $\underline{t}$ is the thickness of ponded water on the surface. The potential at the drain $\emptyset_o$ is equal to $\underline{r}$, the radius of the drain. This boundary condition is obtained from the assumption that the drain is running full with no back pressure. Thus, the potential difference from the top boundary to the drain is equal to $\emptyset_n - \emptyset_o$ or $\underline{d} + \underline{t} - \underline{r}$. If we apply $\underline{V}$ volts to the upper network boundary, the potential $\underline{V}_{\underline{n}}$ at the top boundary is $\underline{V}$. The potential at the drain terminal $\underline{V}_o$ is zero. Thus $(\underline{V}_{\underline{n}} - \underline{V}_o) = \underline{V}$ volts; and $\underline{C}_o$ can be determined by equation 20.

$$\underline{C}_o = \frac{\underline{V}}{\underline{d} + \underline{t} - \underline{r}} \tag{20}$$

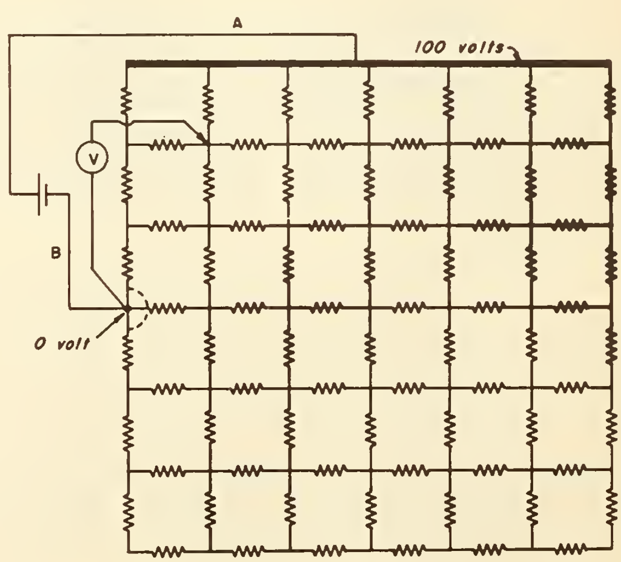



FIGURE 29.—Application of voltage to the network. The positive pole of the battery is connected to the top boundary and the ground is connected to the drain. A vacuum tube voltmeter ($\underline{V}$) is used to measure voltage distribution at each grid point.

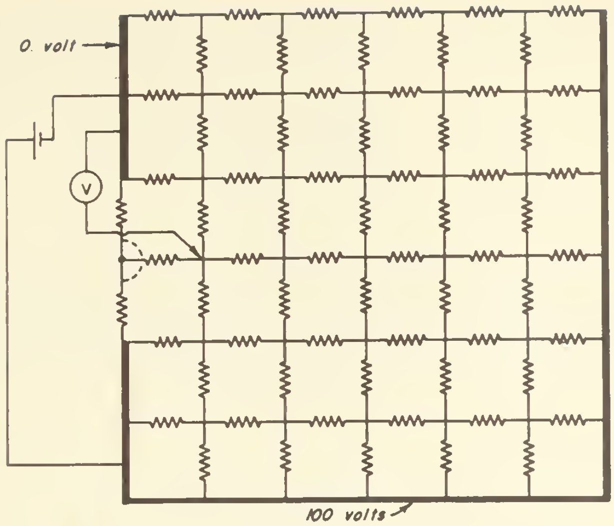



FIGURE 30.—Reversing the boundary of the network to obtain streamlines.

After $\underline{C_o}$ is determined, we then measure the total current $\underline{I}$ passing through the circuit. The flow rate $\underline{Q}$ is then determined by equation 48 in appendix, pt. I.

One disadvantage of the above method is the error and inconvenience involved in measuring the current $\underline{I}$. In the first place, the characteristic resistance $\underline{R_o}$ should be chosen in the range of 5,000 to 10,000 ohms to minimize errors in voltage readings at the network nodes. (See also appendix, pt. III.) This range is chosen so that the resistance of the network will be far less than the internal resistance of the VTVM of 11 megohms and more than the resistance due to wiring and metal contact (i.e., from 1 to 10 ohms). When the characteristic resistance is chosen in this range, the current passing through the network will be too low for accurate measurement. To increase the current reading, one has to increase the applied voltage. This is somewhat dangerous and inconvenient. Also there is the risk of altering the resistance values in the network due to greater current flow.

These disadvantages are avoided by using the following method for determining the flow rate: the total resistance $\underline{R_n}$ between the top boundary of the network and the drain terminal is measured. This is done by attaching wires A and B of figure 29 to an ohmmeter rather than to a battery, or better, to a Wheatstone bridge. The resistance of the network $\underline{R_n}$ can be measured to an accuracy within 0.5 percent with a Wheatstone bridge. The current $\underline{I}$ is given by $\frac{\underline{V}}{\underline{R_n}}$ ; and from equation 20, $\underline{V} = \underline{C_o}\,(\underline{d}+\underline{t}-\underline{r})$. Therefore, the current $\underline{I}$ can be expressed as follows:

$$\underline{I} = \frac{\underline{C_o}(\underline{d} + \underline{t} - \underline{r})}{\underline{R_n}} \tag{21}$$

If we substitute the value of $\underline{I}$ in equation 48 of appendix I and designate $\underline{Q}'$ as the flow rate of the whole drain, we obtain the relationship given by equation 22.

$$\underline{Q}' = \frac{2\underline{K}(\underline{d} + \underline{t} - \underline{r})}{\underline{R}_n} \underline{R}_o \tag{22}$$

The factor 2 shown in equation 22 is to convert the flow rate from that of a half drain to that of a whole drain $\underline{Q}'$. Also from equation 22 we can see that the drain flow rate $\underline{Q}'$ is independent of the applied voltage $\underline{V}$ and the current $\underline{I}$. All we need to know is the resistance $\underline{R}_n$ across the network. The general expression for the flow rate can be expressed as

$$\underline{Q} = \frac{\underline{K}(\phi_n - \phi_o)}{\underline{R}_n} \underline{R}_o \tag{23}$$

where $\phi_n$ is the hydraulic potential at the top boundary and $\phi_o$ is the hydraulic potential at the drain or at the point of discharge.

The drawings shown in figures 29 and 30 are only schematic. For more detail, let us assemble a network to represent a flow problem for the following dimensions: The half spacing between drains (a/2) is 72 feet, and an impervious layer ($\underline{h}$) is 18 feet below the ground surface. The depth to the drain center line ($\underline{d}$) is 12 feet. The drain diameter ($2\underline{r}$) is 4 inches. A water table is maintained at the ground surface, and a hydraulic conductivity ($\underline{K}$) of 0.001 foot/min. is assumed.

Because of symmetry, the network is assembled to represent only half of the flow region. For horizontal distances up to 6 feet from the drain, each linear foot in the soil is represented by one resistive unit (i.e., a 1- by 1-foot mesh). Between 6 and 8 feet, a 2- by 2-foot mesh is used, while a 2- by 8-foot mesh is employed for distances greater than 8 feet. For a 2- by 8-foot mesh, the horizontal distance is 8 feet. A potential difference of 100 volts is applied through the network, the potential at each node is measured, and the equipotential lines are evaluated by interpolating the potential between adjacent nodes (see fig. 31). After reversing the boundaries and again measuring the potentials, the flow lines are determined in a similar manner and are also shown in figure 31.

The characteristic resistance $\underline{R}_o$ is chosen to be 5,000 ohms, and the resistance across the network $\underline{R}_n$ for the drain depth at 12 feet is measured to be 8,797 ohms. Using equation 22, we determine the flow rate $\underline{Q}$ to be

$$\underline{Q} = \frac{2 \times 0.001\ (12+0 - 2/12) \times 5{,}000}{8{,}797}$$

$$= 0.01345\ (\text{ft.}^3/\text{ft. drain-min.})$$

In the case of stratified or anisotropic medium, the flow rate $\underline{Q}$ can be determined the same way as for homogeneous medium. However, we have to keep in mind that the value for the hydraulic conductivity $\underline{K}$ is that chosen to be represented by the characteristic resistance $\underline{R}_o$. For example, consider the case for flow of water through a stratified soil having hydraulic conductivities of 2.24, 0.56 and 0.28 cm./hr., respectively (see section VI). If $\underline{R}_2$ is 5,000 ohms and represents the conductivity $\underline{K}_2$ of 0.56 cm./hr., the values of $\underline{K}_2$ and $\underline{R}_2$ will appear in place of $\underline{K}$ and $\underline{R}_o$, respectively, in equation 22. The value of $\underline{K}_1$ and $\underline{K}_3$ can be ignored, since their overall effect is already included in the resistance across the network $\underline{R}_n$. In figure 32 a flow net for the flow of water through an anisotropic soil is shown. The method for calculating the resistances is given in the section beginning on p. 21.

The results obtained from an electrical resistance network are associated with errors that are due to the mesh size employed, the selection of the characteristic resistance $\underline{R}_o$, and the inaccuracies in setting individual resistors. The errors due to mesh size is discussed by Liebmann (6), whereas those errors caused by the other two sources are discussed in the appendix, part III.

## REPRESENTING AN UNSATURATED REGION OF SOIL

Under saturated flow conditions, the hydraulic conductivity $\underline{K}$ in each layer of soil is independent of its location within the layer. In unsaturated flow, the hydraulic conductivity of a particular layer may differ from place to place. For flow problems encountered during drainage, unsaturated flow generally

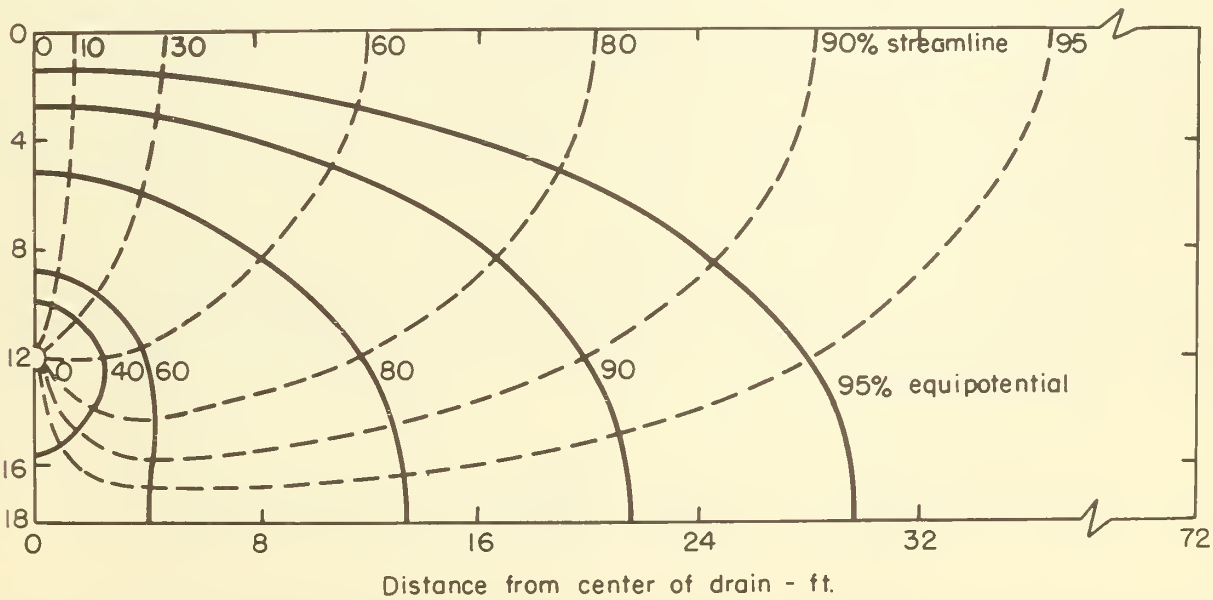



FIGURE 31.—Flow net for a homogeneous, isotropic soil during ponded flow into a 4-inch drain.

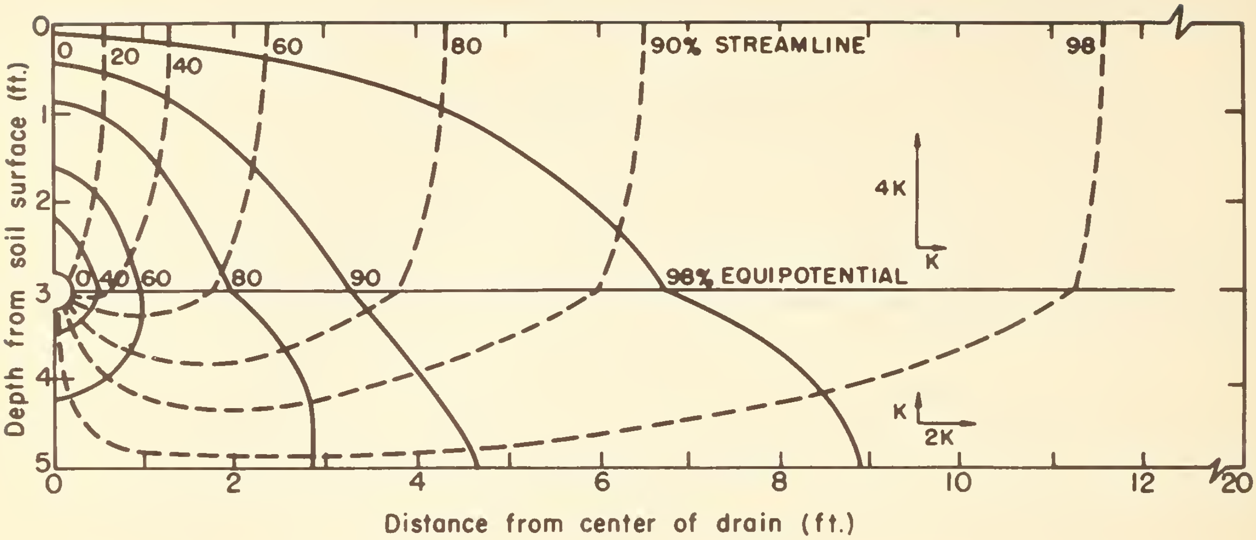



FIGURE 32.—Flow net for a layered, anisotropic soil during ponded flow into a 4-inch drain.

occurs above a nearly horizontal water table whereas saturated flow occurs below this surface.

To represent a block of soil having variable spatial conductivity with a mesh of resistors, first consider a piece of conducting paper with the side dimension $\underline{a}$ by $\underline{b}$ as shown in figure 33. Cartesian coordinates are set up with the $\underline{x}$-axis along the lower edge and the $\underline{y}$-axis along the left boundary. Let us assume that the resistivity $\underline{R}$ is only a function of the vertical coordinate, i.e., $\underline{R} = \underline{R}_{(\underline{y})}$. This condition approximates that of a small block of unsaturated soil in which changes in moisture content (and thus in hydraulic conductivity) in the horizontal direction are quite small, as compared to those in the vertical. Previous work has shown this condition to be essentially the case for unsaturated flow above a falling water table.

If we use the "building block" approach, we must calculate the overall resistance $\underline{R}_V$ in the vertical direction and the overall resistance $\underline{R}_H$ in the horizontal direction. Since the resistivity varies only in the $\underline{y}$ direction, let us then consider a small strip with dimension $\underline{dy}$ by $\underline{a}$. The incremental resistance $\Delta \underline{R}_V$ can be expressed as

$$\Delta \underline{R}_V = \frac{\underline{dy}}{\underline{a}} \underline{R}_{(\underline{y})} \tag{24}$$

From inspection, one can see that the incremental resistances $\Delta R_V$ are joined in series, so that the total resistance in the vertical direction $\underline{R}_V$ is equal to

$\Sigma \Delta R_V$. Therefore,

$$R_V = \frac{1}{a}\int_0^b R_{(y)} dy \tag{25}$$

To determine $R_H$, consider the same strip in figure 33. The incremental resistance $\Delta R_H$ is expressed as

$$\Delta R_H = \frac{a}{dy} R_{(y)} \tag{26}$$

Since the total resistance $R_H$ is the result of joining in parallel the incremental strips of resistance $\Delta R_H$, direct integration of equation 26 is not valid.

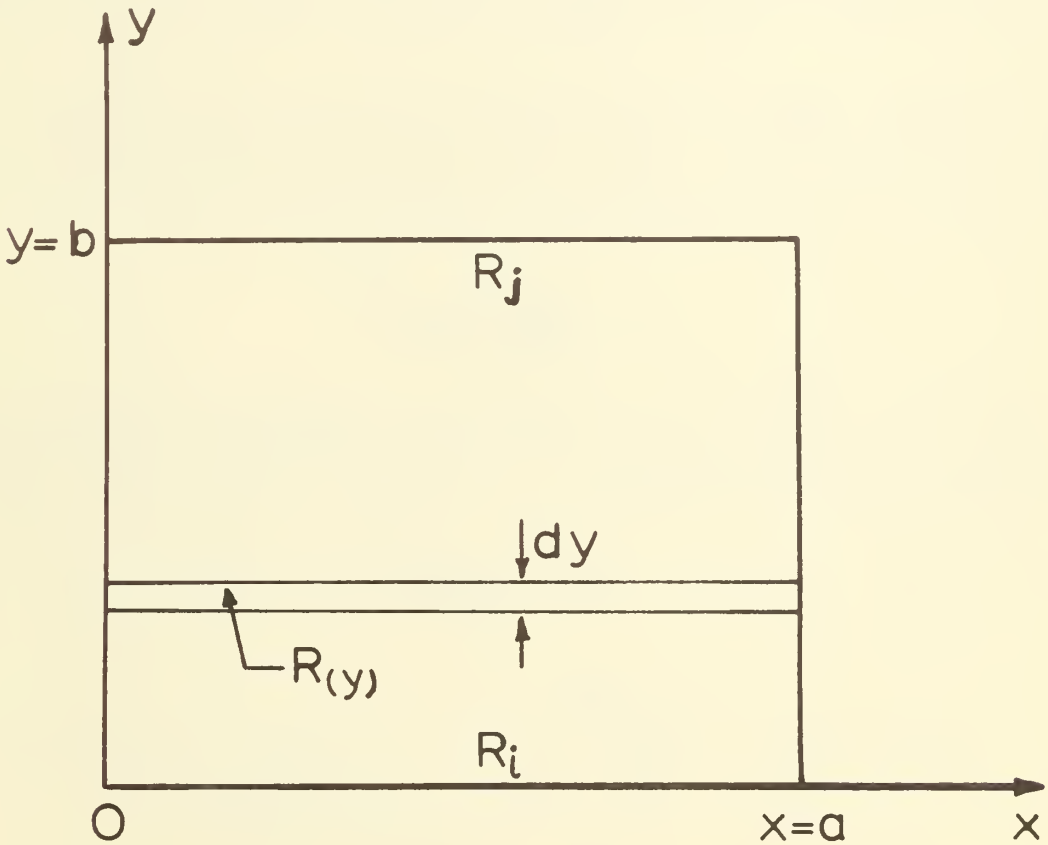



FIGURE 33.—Diagram showing flow region with variable resistivity in the $y$-direction. The resistivity is symbolized by $R_{(y)}$, with values at the upper and lower boundaries given by $R_j$ and $R_i$, respectively.

However, if we designate the term $\Delta G_H$ as the incremental conductance in the horizontal direction and equal to $\frac{1}{\Delta R_H}$, equation 26 can then be expressed as

$$\Delta G_H = \frac{1}{a} \frac{dy}{R_{(y)}} \tag{27}$$

so that

$$G_H = \frac{1}{R_H} = \frac{1}{a}\int_0^b \frac{1}{R_{(y)}}\, dy \tag{28}$$

From the procedure given above, two points should be emphasized: (1) Integration of incremental resistances can be made if they are joined in series; (2) integration of incremental conductances can be made if they are joined in parallel.

To illustrate the use of equations 25 and 28, first assume that the resistivity $R_{(y)}$ varies linearly in the $y$ direction between $y = 0$ and $y = b$. This assumption introduces only a small error, unless the function $R_{(y)}$ is changing rapidly in the vicinity $0 \leq y \leq b$ or the soil block represented is of relatively large dimension, i.e., a coarse mesh is used. Thus,

$$R = R_{(y)} = Ay + B \tag{29}$$

where $A$ and $B$ are constants. From figure 33, designate the resistivity of the lower edge as $R_i$ and of the upper edge as $R_j$. (The determination of $R_i$ and $R_j$ will be discussed in a later section.) By setting first $y = 0$ and $y = b$, the constants $A$ and $B$ can be determined as

$$A = \frac{R_j - R_i}{b} \tag{30}$$

$$B = R_i \tag{31}$$

Therefore,

$$R = R_{(y)} = \left(\frac{R_j - R_i}{b}\right) y + R_i \tag{32}$$

Substituting $R(y)$ of equation 32 in equations 25 and 28 and carrying out the integrations, we find that

$$R_V = \frac{b}{a}\left(\frac{R_j + R_i}{2}\right) \tag{33}$$

$$R_H = \frac{a}{b}\,\frac{(R_j - R_i)}{\log_e \dfrac{R_j}{R_i}} \tag{34}$$

To test the validity of equations 33 and 34, let us consider the case when the entire flow region is saturated. In such case $R_i$ and $R_j$ will be equal to $R_o$, the characteristic resistance of the square piece of resistive paper having uniform resistivity $R_o$. We then need to prove that under such conditions $R_V$ and $R_H$ also be equal to $\frac{b}{a}R_o$ and $\frac{a}{b}R_o$, respectively. It is obvious from equation 33 that if $R_j$ and $R_i$ are replaced by $R_o$ then $R_V = \frac{b}{a}R_o$. The same substitution, however, will not yield direct proof in the case of $R_H$. For if $R_j$ and $R_i$ are replaced by $R_o$ in equation 34, we would have the indeterminate form $\frac{0}{0}$. With the help of equations 30 and 31, equation 34 can be rewritten as

$$R_H = \frac{aA}{\log_e\left(\dfrac{bA}{B} + 1\right)} \tag{35}$$

If $R_j = R_i = R_o$, then from equations 30 and 31 we would have $A = 0$ and $B = R_o$. If the values of $A$ and $B$ are substituted into equation 35, $R_H$ will be equal to $\frac{0}{0}$; and L'Hospital's Rule is applicable with respect to $A$. Therefore, differentiations of the numerator and the denominator of equation 35 with respect to $A$ yields

$$R_H = \lim_{A\to 0}\left[\frac{aA}{\log_e\left(\dfrac{bA}{B}+1\right)}\right] = \lim_{A\to 0}\left[\frac{a\left(\dfrac{bA}{B}+1\right)}{\dfrac{b}{B}}\right] = \frac{a}{b}B = \frac{a}{b}R_o \tag{36}$$

It should be noticed that the expression $R_H = \frac{a}{b}B$ in equation 36 is valid both in the saturated and unsaturated zones, as long as $R_j = R_i = B$. Accordingly,

in the saturated zone $B$ is equal to $R_o$, whereas in the unsaturated zone $B$ is equal to either $R_j$ or $R_i$.

After $R_V$ and $R_H$ are determined, we can now proceed to represent the block of soil or resistive paper shown in figure 33 with four resistors. This representation is shown in figures 34, a and b. The terms $R_V$ and $R_H$ are calculated from equations 33 and 34, respectively.

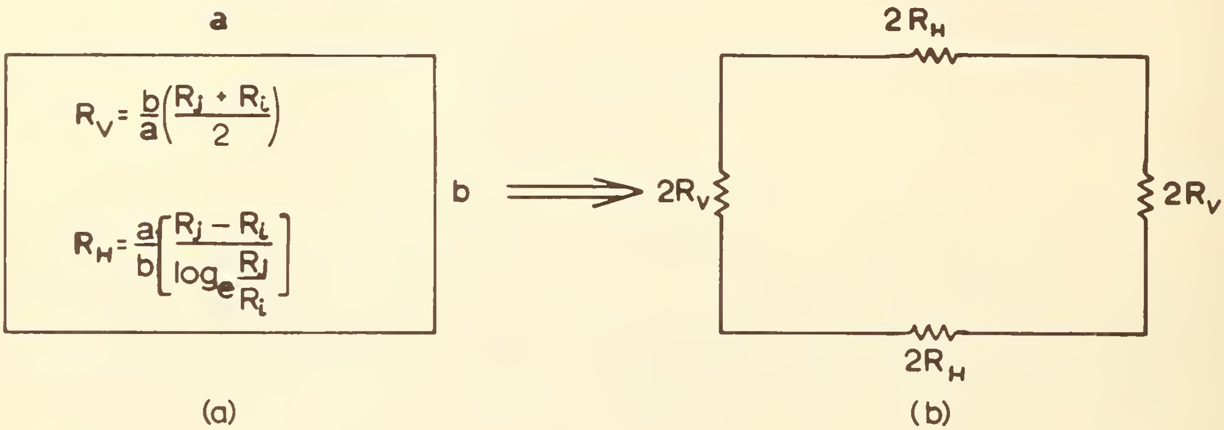



FIGURE 34.—Representing the flow region shown in (a) with a mesh of 4 resistors.

From equations 33 and 34, which yield the values of $R_V$ and $R_H$, we have two unknowns to determine -- $R_i$ and $R_j$. These terms represent the resistivity of the lower and upper boundaries, respectively, of the nonuniform conducting paper shown in figure 33. To evaluate $R_i$ and $R_j$ for the study of flow in unsaturated soil, it is necessary to relate the resistivity of the conducting paper $R$ and the hydraulic conductivity of the soil $K$.

In unsaturated flow, the hydraulic conductivity is dependent on the moisture content of the medium. As the moisture content decreases below saturation, the hydraulic conductivity likewise decreases. Also, as the moisture content decreases, the hydrostatic pressure $h$ of the soil water films increases in the negative direction. In other words, the soil moisture tension is increased. The unsaturated hydraulic conductivity is, in turn, related to the hydrostatic pressure. The relationship between conductivity and hydrostatic pressure can be obtained by methods described by Childs and Collis-George (2) and by Neilsen and coworkers (10). An example[6] of this type of relationship is shown in figure 35. From the graph one can see that each type of medium has a characteristic relationship between the hydraulic conductivity and the hydrostatic pressure.

To relate the resistivity $R$ to the hydrostatic pressure $h$, we first use the relationship between the resistivity $R$ of the conducting paper and the hydraulic conductivity $K$ of the fluid flow medium. This can be written as $K = \frac{c}{R}$ where $c$

[6]Thiel, T. J. Unpublished Master's Thesis, Ohio State University, Columbus, Ohio 1959.

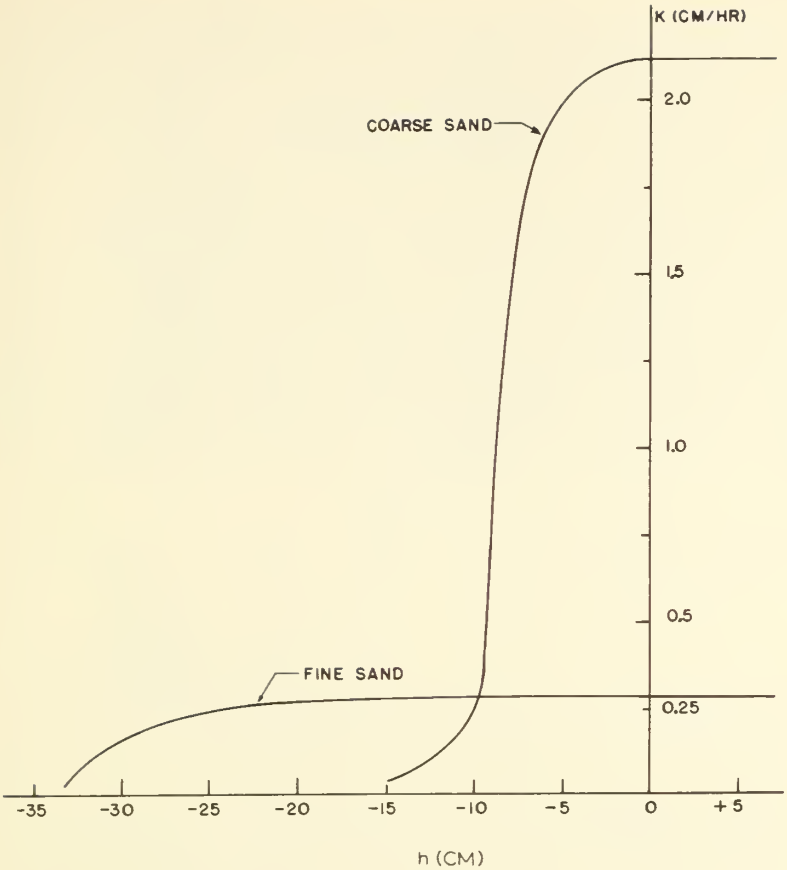



FIGURE 35.—Fluid conductivities $\underline{K}$ of fine and coarse sands as a function of hydrostatic pressure $\underline{h}$. The points plotted on and to the right of the ordinate represent the saturated fluid conductivities of the two sand sizes.

is a constant relating $\underline{K}$ to $\underline{R}$. From this relationship we can then write equation 37.

$$\frac{\underline{R}_0}{\underline{R}} = \frac{\underline{K}}{\underline{K}_0} \tag{37}$$

$\underline{K}_0$ and $\underline{K}$ are the hydraulic conductivities, respectively, of saturated and unsaturated soil; and $\underline{R}_0$ and $\underline{R}$ are the electrical resistivities, respectively, of uniform (analogous to saturated soil) and nonuniform conducting medium (analogous to unsaturated soil). Both $\underline{K}$ and $\underline{R}$ are space functions in their respective media. Another source of needed information is the equation relating the potential head $\emptyset$, the elevation head $\underline{y}$, and the hydrostatic pressure head $\underline{h}$. These three are related by equation 38 for cases where only hydrostatic pressure and elevation components of potential are significant.

$$\emptyset = \underline{y} + \underline{h} \tag{38}$$

With the information above, we are now able to utilize the network analog to solve unsaturated steady-state flow problems. To illustrate, consider a soil section having a dimension of 10 by 10 feet, as shown in figure 36. On the

right and left side of the soil section, the hydraulic heads are kept constant at 8 and 6 feet, respectively.[7] For simplicity, assume that the characteristic relationship between $\underline{K}/\underline{K}_o$ and the hydrostatic pressure $\underline{h}$ is as shown in figure 37. From figure 37, we have $\underline{K}/\underline{K}_o = 1$ for $\underline{h} \geq -1.0$, $\underline{K}/\underline{K}_o = 0.5\underline{h} + 1.5$ for $-2.8 \leq \underline{h} \leq -1$, and $\underline{K}/\underline{K}_o = 0.1$ for $\underline{h} \leq -2.8$. The same problem can be assembled on the network with each mesh of resistors representing a small square block of soil of dimension 1 by 1 foot. The grid points are shown in figure 36. Let us first assemble the network to represent a saturated and homogeneous soil medium having resistances $\underline{R}_o$ everywhere in the interior and $2\underline{R}_o$ along the boundaries. Eight boundary resistors on the right side and six on the left side are shorted out as shown in the figure, and potentials of 8 and 6 volts are applied along the right and left boundaries, respectively. The potential at each grid point is measured and recorded. The magnitude of $\underline{h}$ at each grid point is then calculated by equation 38. From the values of $\underline{h}$, $\underline{R}_i$ and $\underline{R}_j$ can be

[7] A similar type of study had been reported by Luthin and Day (8).

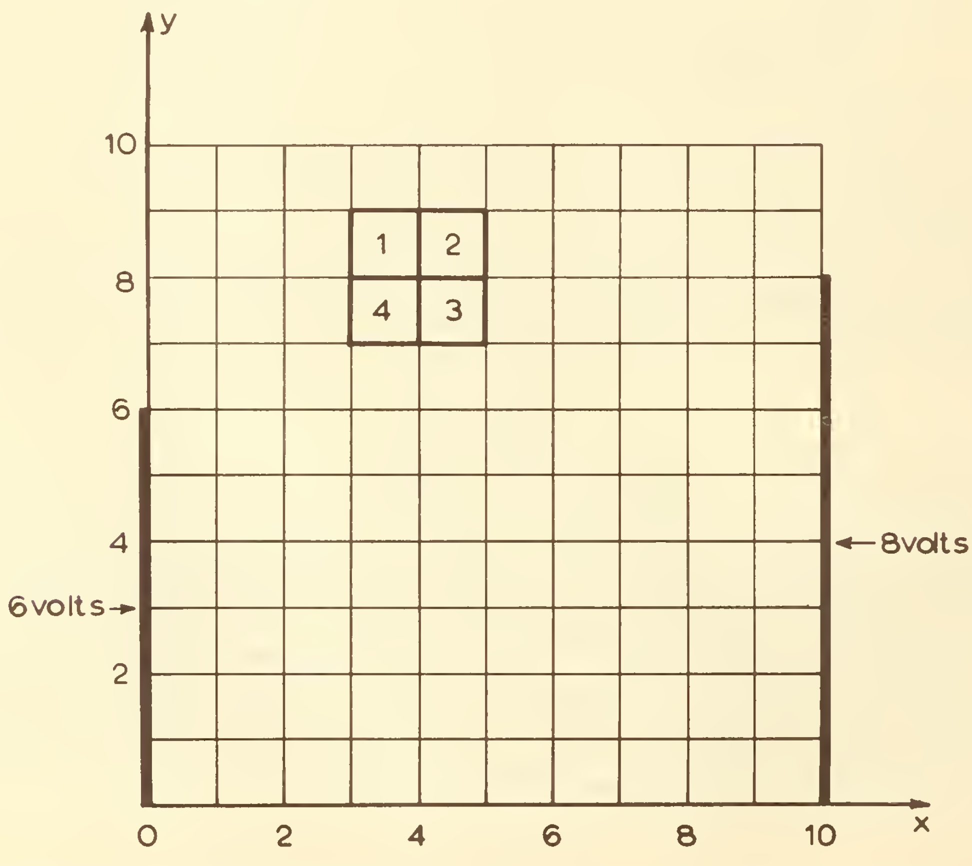



FIGURE 36.—Network representing a 10' x 10' section of soil. Each square mesh has a dimension of 1' x 1'. Hydraulic head potentials of 6 and 8 feet are kept along the left and right boundaries, respectively. These boundary potentials are obtained by applying an e.m.f. of 6 and 8 volts along the left and right boundaries of the network.

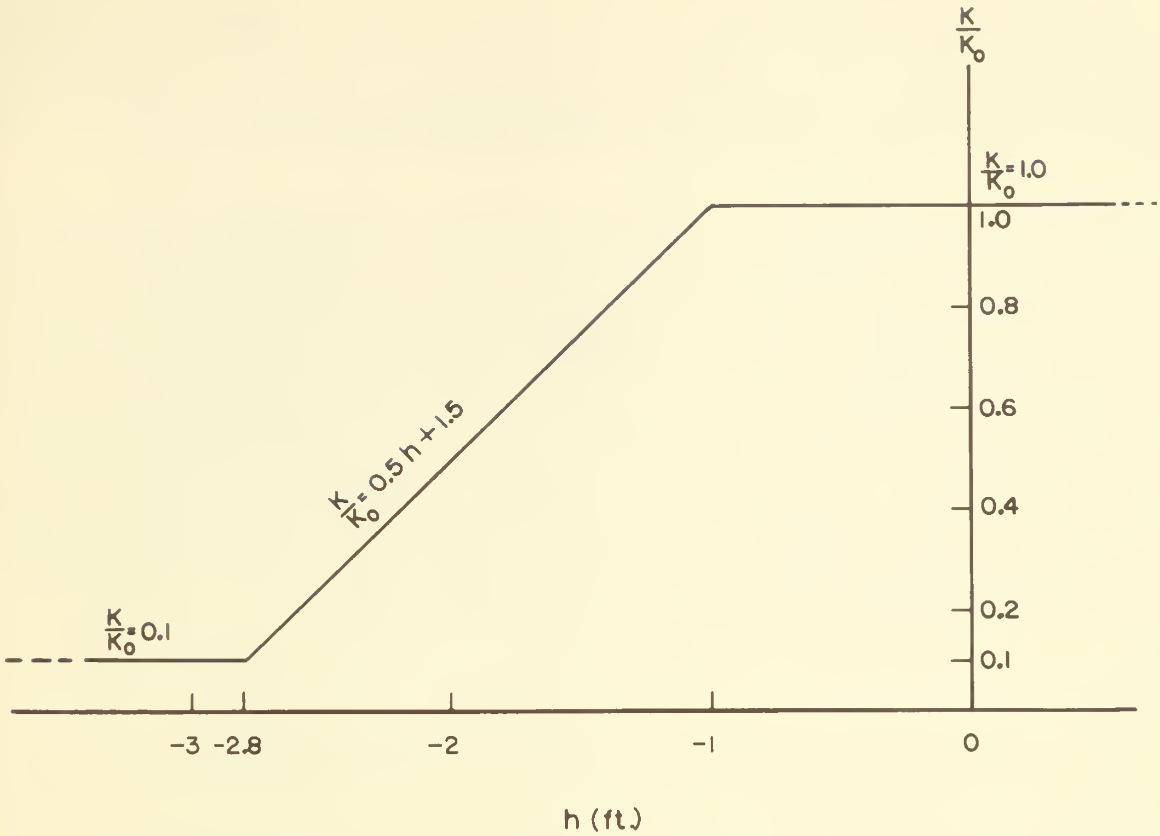



FIGURE 37.—Illustrative curve, showing the relationship of $K/K_o$ and hydrostatic pressure h. The saturated conductivity is given by $K_o$.

obtained from the relationship given in figure 37. The resistances of the unsaturated medium can then be calculated as illustrated by the diagram shown in figure 38. The section of soil represented in figure 38 is that shown with a heavy line in the interior of figure 36.

To calculate the resistances, consider the section in figure 38, a, with the potentials $\phi$ recorded at each node. As explained above, the initial values of the potential are obtained by utilizing a homogeneous and saturated flow medium. The next step is to adjust the resistors in accordance with the magnitude of the hydrostatic pressure h. To proceed, the section in figure 38, a is first separated into four small squares (see fig. 38, b). The potential at each corner is also shown in figure 38, b. By knowing the potential $\phi$ and the elevation from the datum plane y, the hydrostatic pressure can be calculated. For example, the potential at the upper left corner of block No. 1 is 6.86 feet and y is 9 feet. Therefore, the hydrostatic pressure h is 6.86 - 9 feet = -2.14 feet as shown in figure 38, c. From the known value of h, the ratio of conductivity $K/K_o$ can be obtained from figure 37 and is found to be 0.43. The same procedure is applicable for obtaining $K/K_o$ when the experimental curve shown in figure 35 is used in the network study. However, if the problem is

solved by numerical analysis and a high speed electronic computer is used, the relationship between $K/K_o$ and $h$ must be expressed in an analytical equation, the latter sometimes being somewhat cumbersome to use. This difficulty can be avoided by approximating the curve shown in figure 35 with a series of broken straight lines for which linear equation can be assigned to each line.

After the value of $K/K_o$ is obtained, the value of $R/R_o$ can be calculated from equation 37 as $1/0.43 = 2.33$ and is shown in figure 38, d. With the same procedure, the value of $R/R_o$ at the upper right corner of block No. 1 is calculated to be 2.04 (figure 38, d), and the value of the two are averaged to yield 2.19. This value is then set equal to $R_{j1}/R_o$ so that $R_{j1} = 2.19\ R_o$, as shown in figure 38, e. The magnitude of $R_{i1}$ of block No. 1 can be calculated in the same manner starting from the values of $\emptyset$ at the lower corners. From $R_{i1}$ and $R_{j1}$, the values of $R_{V1}$ and $R_{H1}$ of block No. 1 can be calculated with equations 33 and 34 and are found to be 1.62 $R_o$ and 1.55 $R_o$, respectively, as shown in figure 38, f. The network of four resistors representing block No. 1 is shown in figure 38, g. The same procedure is carried out for blocks Nos. 2, 3, and 4, and they are joined together in parallel, as shown in figure 38, h. The same procedure is carried on throughout the unsaturated region of soil.

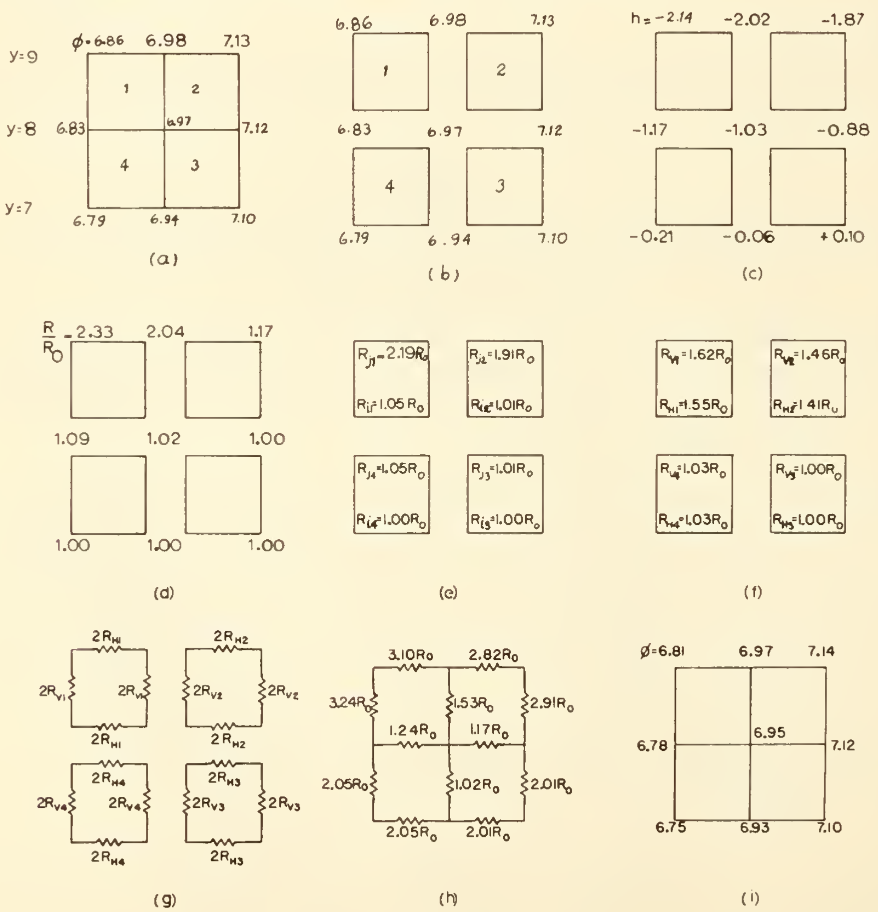



FIGURE 38.—Diagrams illustrating calculation of the resistances which represent unsaturated flow medium. See text for detailed discussion.

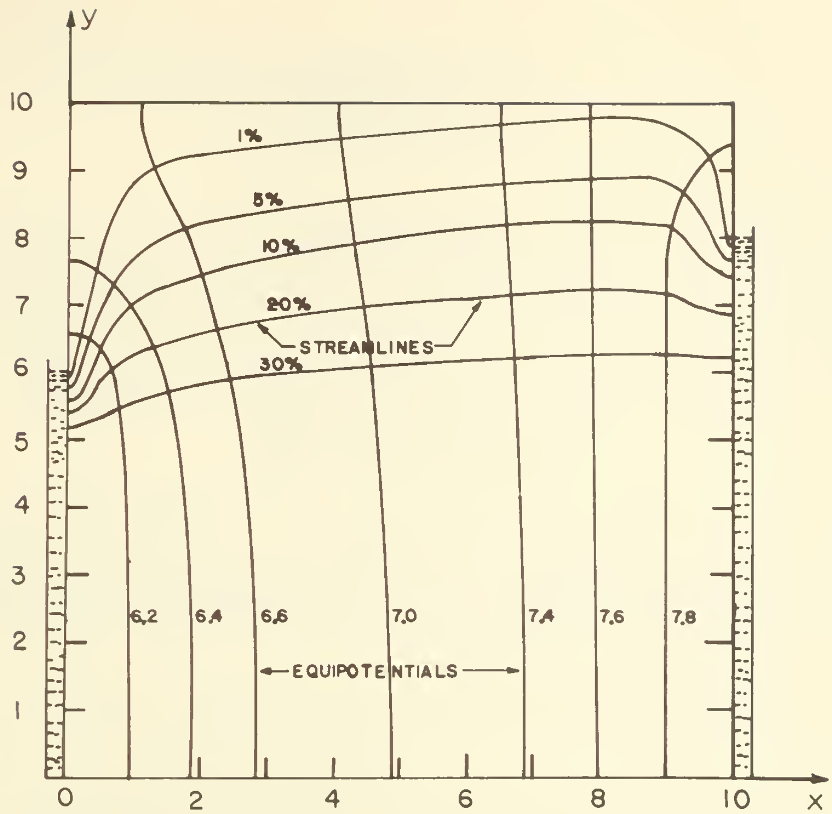



FIGURE 39.—Flow net for steady-state flow of water. Water is ponded at left and right boundaries, and the relationship between $\underline{K}$ and $\underline{h}$ is that shown in figure 37. The procedure illustrated in figure 36 and figure 38 is followed on the analysis.

At this point the voltages are then reapplied to the network, and the potential distribution is again recorded. The new values of node potential are shown in figure 38, i. One can see that the potential distribution of the new set is different from the previous one. The entire procedure is repeated until no significant potential change is observed, and the relaxation is completed. The flow net for this particular problem is shown in figure 39.

The discussion above deals only with an unsaturated medium whose conductivity varies only in the vertical direction. If, in some cases, the conductivity varies in both vertical and horizontal directions, the determination of $\underline{R}_V$ and $\underline{R}_H$ can only be obtained approximately. It would appear, however, that acceptable accuracy can be obtained by using successive linear approximations of resistivity $\underline{R}$. On this basis the expressions $\underline{R}_V$ and $\underline{R}_H$ for a block of soil having dimensions $\underline{a}$ by $\underline{b}$ can be approximated as

$$\underline{R}_V = \frac{\underline{b}}{\underline{a}}\left(\frac{\underline{R}_j + \underline{R}_i}{2}\right) \tag{39}$$

and

$$\underline{R}_{\underline{H}} = \frac{\underline{a}}{\underline{b}}\left(\frac{\underline{R}_{\underline{j}} + \underline{R}_{\underline{i}}}{2}\right) \tag{40}$$

where $\underline{R}_{i}$ and $\underline{R}_{j}$ are determined similar to those illustrated in figure 38. This approximation is based on the assumption that the unsaturated soil is homogeneous with respect to the conductivities in the horizontal and vertical direction.

## LITERATURE CITED


(1) Bouwer, H., and Little, W. C.
1959. A unifying numerical solution for two-dimensional steady flow problems in porous media with an electrical resistance network. Soil Sci. Soc. Amer. Proc. 23:91-96.

(2) Childs, E. C. and Collis-George, N.
1950. The permeability of porous materials. Roy Soc. London, Proc., ser. A, 201:392-405.

(3) Kirkham, D.
1949. Flow of ponded water into drain tubes in soil overlying an impervious layer. Amer. Geophysics. Union Trans. 30:369-385.

(4) Kirkham, D. and Gaskell, R. E.
1950. The falling water table in tile and ditch drainage. Soil Sci. Soc. Amer. Proc. 15:37-42.

(5) Kraus, J. D.
1953. Electromagnetics. pp. 426-429. McGraw-Hill Book Co., New York.

(6) Liebmann, G.
1950. Solution of partial differential equations with a resistance network analogue. Brit. Jour. Appl. Phys. 1:92-103.

(7) Luthin, J. N.
1950. An electrical resistance network solving drainage problems. Soil Sci. 75:259-275.

(8) Luthin, J. N., and Day, P. R.
1955. Lateral flow above a sloping water table. Soil Sci. Soc. Amer. Proc. 19:406-410.

(9) Luthin, J. N., and Gaskell, R. E.
1950. Numerical solutions for tile drainage of layered soils. Amer. Geophysics. Union Trans. 31:595-602.

(10) Neilsen, D. R., Kirkham, Don, and Perrier, E. R.
1960. Soil capillary conductivity: Comparison of measured and calculated values. Soil Sci. Soc. Amer. Proc. 24:157-160.

(11) Vimoke, B. S., Tyra, T. D., Thiel, T. J., and Taylor, G. S.
1962. Improvements in construction and use of resistance networks for studying drainage problems. Soil Sci. Soc. Amer. Proc. 2:203-207.

(12) Westman, H. P., Ed.
1959. Reference data for radio engineers. Ed. 4, 590 pp. International Telephone & Telegraph Corp., New York.

# APPENDIX

## I. REPRESENTING WATER FLOW IN SOILS WITH ELECTRICAL CURRENT FLOW IN CONDUCTING PAPER

A square of conducting paper with dimensions $\underline{a}$ by $\underline{a}$ can be used to represent any square dimension of uniform soil which has unit thickness. For example, consider the block of saturated and homogeneous soil of hydraulic conductivity $\underline{K}$ shown in figure 40, a. Assume that constant potentials $\emptyset_n$ and $\emptyset_o$, respectively, are applied to the upper and lower surfaces. A steady state

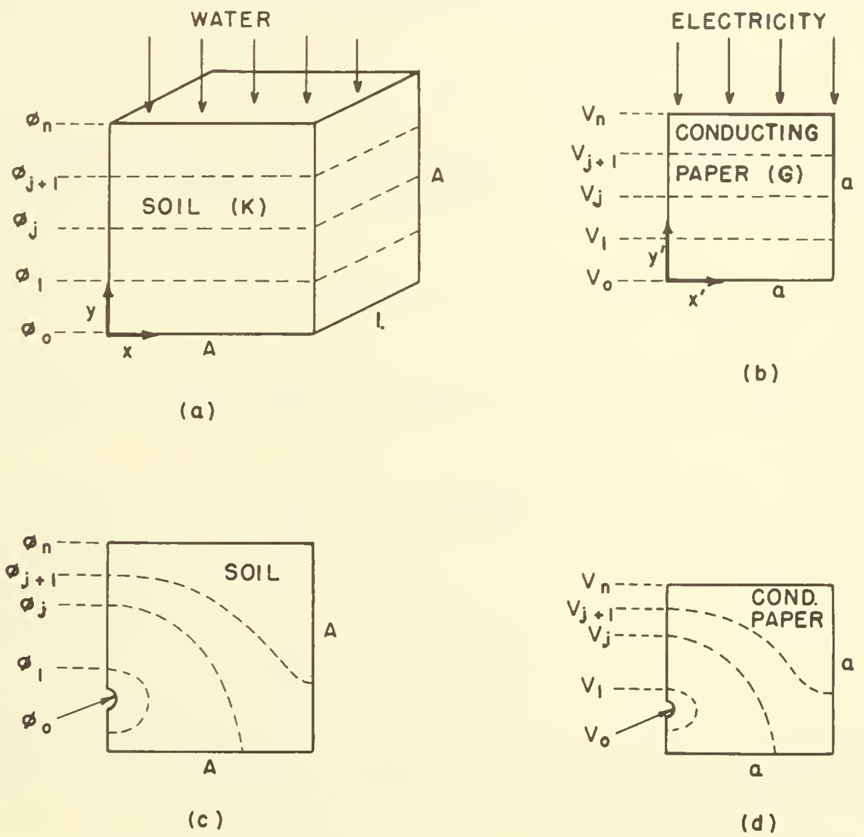



FIGURE 40.—Diagram illustrating the analogy of water flow through soil and electrical flow by conducting paper. One-dimensional flow is illustrated in (a) and (b); and 2-dimensional flow in (c) and (d).

flow will be established, and the quantity of water $\underline{Q}$ leaving the lower surface per unit time will be given by equation 41.

$$\underline{Q} = \underline{v}\ \underline{A}\ \underline{l} = -\underline{K}\frac{d\phi}{d\underline{y}}\ \underline{A} \tag{41}$$

where $\underline{v}$ is the average flow velocity per unit area and $\underline{A}$ is the dimension of the square block in the $\underline{x}$-direction. Similarly, the current $\underline{I}$ leaving the lower edge of the conducting paper shown in figure 40, b will be given by the following relationship:

$$\underline{I} = \underline{i}\ \underline{a} = -\underline{G}\frac{d\underline{V}}{d\underline{y}'}\ \underline{a} \tag{42}$$

where $\underline{G}$ is the electrical conductivity, $\underline{i}$ is the current flux per unit length in the $\underline{x}'$-direction, and $\underline{V}$ is the voltage.

Since both media are homogeneous with respect to their conductivity, the functions $\phi$ and $\underline{V}$ will be related in the following manner:

$$\underline{C}_o(\phi_{\underline{j}+1} - \phi_{\underline{j}}) = (\underline{V}_{\underline{j}+1} - \underline{V}_{\underline{j}}) \tag{43}$$

where $\underline{C}_o$ is a proportionality constant. By evaluating equation 43 for $(\underline{j} + 1) = \underline{n}$ and $\underline{j} = 0$, it can be seen that the voltage $\underline{V}$ is related to $\phi$ by equations 44 and 45.

$$\underline{C}_o = (\underline{V}_{\underline{n}} - \underline{V}_o)/(\phi_{\underline{n}} - \phi_o) \tag{44}$$

$$\underline{V}_{(\underline{x}',\underline{y}')} = \underline{C}_o\,\phi_{(\underline{x},\underline{y})} + \underline{C}_1$$
$$\text{where } \underline{x}' = \underline{x}\ \frac{\underline{a}}{\underline{A}}$$
$$\underline{y}' = \underline{y}\ \frac{\underline{a}}{\underline{A}} \tag{45}$$
$$\underline{C}_1 = (\underline{V}_o - \underline{C}_o\phi_o)$$

For the examples shown in figures 40, a and b, it can be seen by inspection that both $\phi$ and $\underline{V}$ vary only in the vertical direction. Thus, if equation 45 is differentiated with respect to $\underline{y}$, the following results:

$$\frac{d\underline{V}}{d\underline{y}'}\left(\frac{\underline{a}}{\underline{A}}\right) = \underline{C}_o\frac{d\phi}{d\underline{y}} \tag{46}$$

By substituting equation 46 in equation 42 and then dividing the resulting expression into equation 41, one obtains the following relationships among the conductivities and flow rates in the two media:

$$\underline{Q} = \frac{\underline{I}\,\underline{K}}{\underline{C}_o\underline{G}} \tag{47}$$

$$\underline{Q} = \frac{\underline{I}\,\underline{K}\,\underline{R}_o}{\underline{C}_o}, \tag{48}$$

$$\text{where } \underline{R}_o = \frac{1}{\underline{G}}$$

From the above analysis, it can be seen that a square of electrical conducting paper can represent one-dimensional flow of water in any square dimension of uniform soil having unit thickness. The potential $\emptyset$ is related to the voltage $\underline{V}$ as shown in equation 45, and the flow rate $\underline{Q}$ is given in terms of $\underline{G}$, $\underline{K}$, and $\underline{I}$ by equations 47 and 48. Although the example presented here is for one-dimensional flow, equation 45 and 47 are equally valid for two-dimensional problems. Examples of the latter are illustrated in figures 40, c and d.

An approach similar to that used to derive equations 45 and 47 can be followed to show that any rectangular piece of conducting paper of dimension $\underline{a}$ by $\underline{b}$ can represent any rectangular block of soil of dimension $\underline{A}$ by $\underline{B}$. Expressions similar to equations 45 and 47 can thus be derived which contain the parameters $\underline{a}$, $\underline{b}$, $\underline{A}$ and $\underline{B}$.

## II. STEPS TO DETERMINE THE POTENTIAL AT A NODE IN NUMERICAL ANALYSIS OR IN A RESISTANCE NETWORK

One of the more diverse cases encountered in determining the potential at a node in saturated, isotropic soil occurs when the node is (1) located at a point on the interface of horizontal layers with different hydraulic conductivity $\underline{K}$ and (2) is surrounded by "building blocks" of different sizes. Figure 41 shows the location of such a node. From figure 41, a, four blocks of soils have dimensions $\underline{a}$ by $\underline{b}$, $\underline{b}$ by $\underline{c}$, $\underline{a}$ by $\underline{d}$, and $\underline{d}$ by $\underline{c}$, respectively. The two blocks at the top have a conductivity $\underline{K}_T$ and the two blocks at the bottom have a conductivity $\underline{K}_B$. The ratio of $\underline{K}_T/\underline{K}_B$ is given by $\underline{m}$. If $\underline{R}_o$ is the characteristic resistance of the upper blocks, the four blocks can be represented by four meshes of resistances as shown in figure 41, b. The parallel sides of the mesh can be joined together as shown in figure 41, c and the final combination is shown in figure 41, d. In figure 41, d, $\underline{R}_1$ is the combined resistance of $\underline{R}_{AB}$ and $\underline{R}_{CD}$, $\underline{R}_2$ is that of $\underline{R}_{DM}$ and $\underline{R}_{GN}$, etc. If $\underline{V}_1$ is the potential at node $\underline{AC}$, $\underline{V}_2$ is the potential at node $\underline{MN}$, etc., then $\underline{V}_o$ is the potential at node $\underline{BDGE}$.

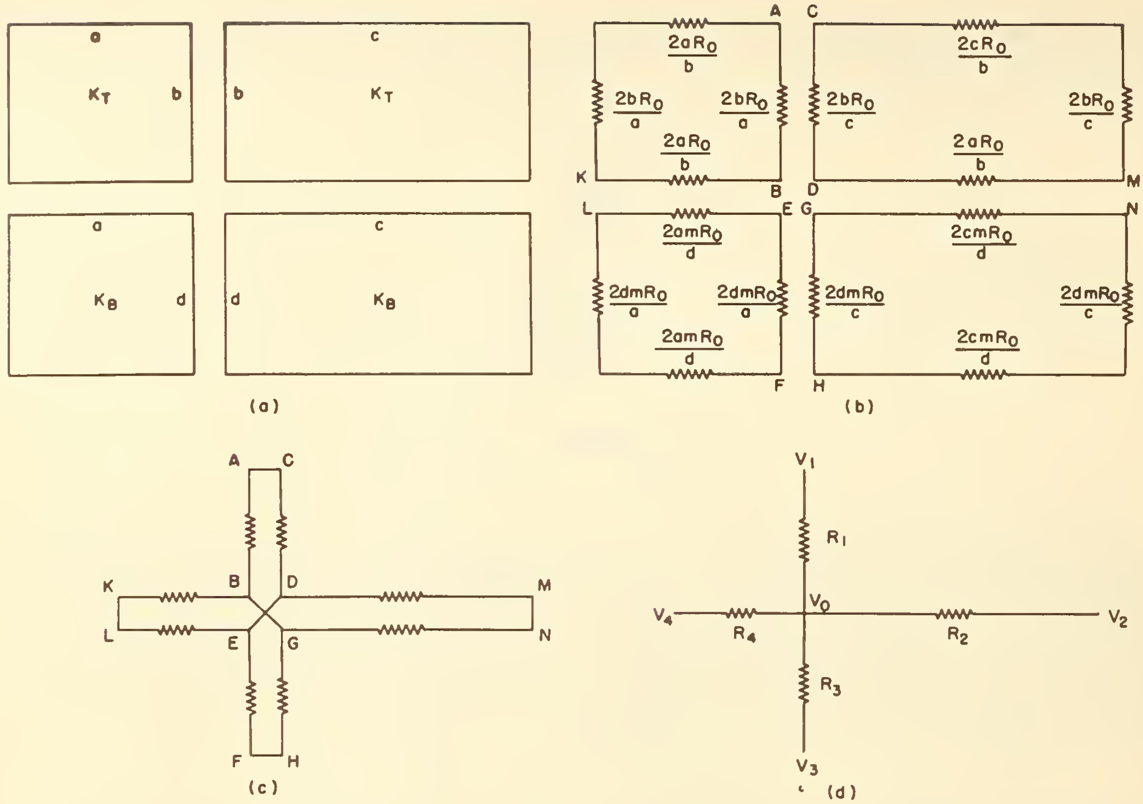



FIGURE 41.—Building block approach used to calculate the potential $V_0$ in relation to the voltages at adjacent nodes, the sizes of rectangular blocks surrounding the node of potential $V_0$, and hydraulic conductivities of the blocks. (See text in appendix, part II, for discussion.)

The potential $V_0$ is now to be determined. Using Kirchhoff's law, we have the following:

$$\frac{V_1 - V_0}{R_1} + \frac{V_2 - V_0}{R_2} + \frac{V_3 - V_0}{R_3} + \frac{V_4 - V_0}{R_4} = 0 \tag{49}$$

Now substitute $R_1$, $R_2$, $R_3$, $R_4$ into equation 49 and solve for $V_0$. We then have

$$\text{since} \quad R_1 = \frac{2bR_0}{(a + c)}, \qquad R_2 = \frac{2mcR_0}{(mb + d)}$$

$$R_3 = \frac{2mdR_0}{(a + c)}, \qquad R_4 = \frac{2maR_0}{(mb + d)} \tag{50}$$

$$V_0 = \frac{macd(a + c)V_1 + abd(mb + d)V_2 + abc(a + c)V_3 + bcd(mb + d)V_4}{macd(a + c) + abd(mb + d) + abc(a + c) + bcd(mb + d)}$$

If $\underline{V}_O$ is within the layer where $\underline{K}_T = \underline{K}_B$ so that $\underline{m} = 1$, and also if $\underline{a} = \underline{c}$, $\underline{b} = \underline{d}$, and $\underline{a}/\underline{b} = \underline{r}$, we will have the following expression:

$$\underline{V}_O = \frac{(\underline{V}_1 + \underline{V}_3) + \left(\frac{1}{\underline{r}^2}\right)(\underline{V}_2 + \underline{V}_4)}{2\left(1 + \frac{1}{\underline{r}^2}\right)} \tag{51}$$

which agrees with results previously reported elsewhere (4, equation 8, p. 39). Finally, if $\underline{a} = \underline{b}$ so that $\underline{r} = \underline{1}$, equation 51 reduces to equation 52

$$\underline{V}_O = \frac{\underline{V}_1 + \underline{V}_2 + \underline{V}_3 + \underline{V}_4}{4}, \tag{52}$$

which is the same as equation 2.

The use of Kirchhoff's law to determine the potential at a node as described above is also applicable in the case of unsaturated steady state flow. As an example, let us first refer to figure 38: By designating the potential of 6.95 volts at the center node of the network fig. 38, i, as $\underline{V}_O$, one will find that this value can also be obtained by Kirchhoff's law. This is done by designating the four surrounding potentials of 6.97, 7.12, 6.93, and 6.78 volts as $\underline{V}_1$, $\underline{V}_2$, $\underline{V}_3$, and $\underline{V}_4$, respectively, and likewise by designating the resistances of 1.53 $\underline{R}_O$, 1.17 $\underline{R}_O$, 1.02 $\underline{R}_O$, and 1.24 $\underline{R}_O$ as $\underline{R}_1$, $\underline{R}_2$, $\underline{R}_3$, and $\underline{R}_4$, respectively (see fig. 38, h). By substituting these values into equation 49 and solving for $\underline{V}_O$, one will find that the value $\underline{V}_O$ is 6.95 volts which is the same as the voltage obtained by measurement.

## Potential at the node adjacent to the drain

In general, the building blocks around the drain region are square.[8] Let us consider the case when the drain is buried in a stratified soil at an interface as shown in figure 42, a. The upper half of the drain is in a layer of conductivity $\underline{K}_T$ and the bottom half of the drain is in a layer of conductivity $\underline{K}_B$. The problem can be solved with the same procedure described above. The only difference is the addition of the term $\underline{C}_d$ which is reported in table 1. The network representing each block of soil is shown in figure 42, b, where $\underline{m}$ is the ratio $\underline{K}_T/\underline{K}_B$. The voltage to be determined is $\underline{V}_O$, which is shown in the center of figure 42, c. Using the same procedure described above, the voltage $\underline{V}_O$ can be determined as shown in equation 53.

$$\underline{V}_O = \frac{2\underline{m}\underline{C}_d\underline{V}_1 + \underline{C}_d(\underline{m} + 1)\underline{V}_2 + 2\underline{C}_d\underline{V}_3 + 2(\underline{m} + 1)\underline{V}_4}{(\underline{m} + 1)(3\underline{C}_d + 2)} \tag{53}$$

[8] Rectangular blocks of soil around the drain region can be handled by proper manipulation of the equation in (12, p. 593, type of line T). This is done by dividing $\underline{Z}_O$ by 2 and changing the term $\underline{D}$ to $\underline{W}/2$.

In the case when the drain is buried within a layer so that $m = 1$, the expression for the potential adjacent to the drain is given by equation 54.

$$V_0 = \frac{V_1 + V_2 + V_3 + \frac{2}{C_d} V_4}{3 + \frac{2}{C_d}} \tag{54}$$

This equation is of similar form to Luthin's equation 4 in reference (8), where the term $h$ in Luthin's equation 4 is equivalent to our $\frac{C_d}{2}$. However, Luthin's expression is based on a linear change in the voltage between the drain and an adjacent node; whereas, equations 53 and 54 assume a logarithmic relationship. It is shown in another report (11) that a logarithmic relationship is necessary to give satisfactory agreement between results obtained with the network and analytical solutions.

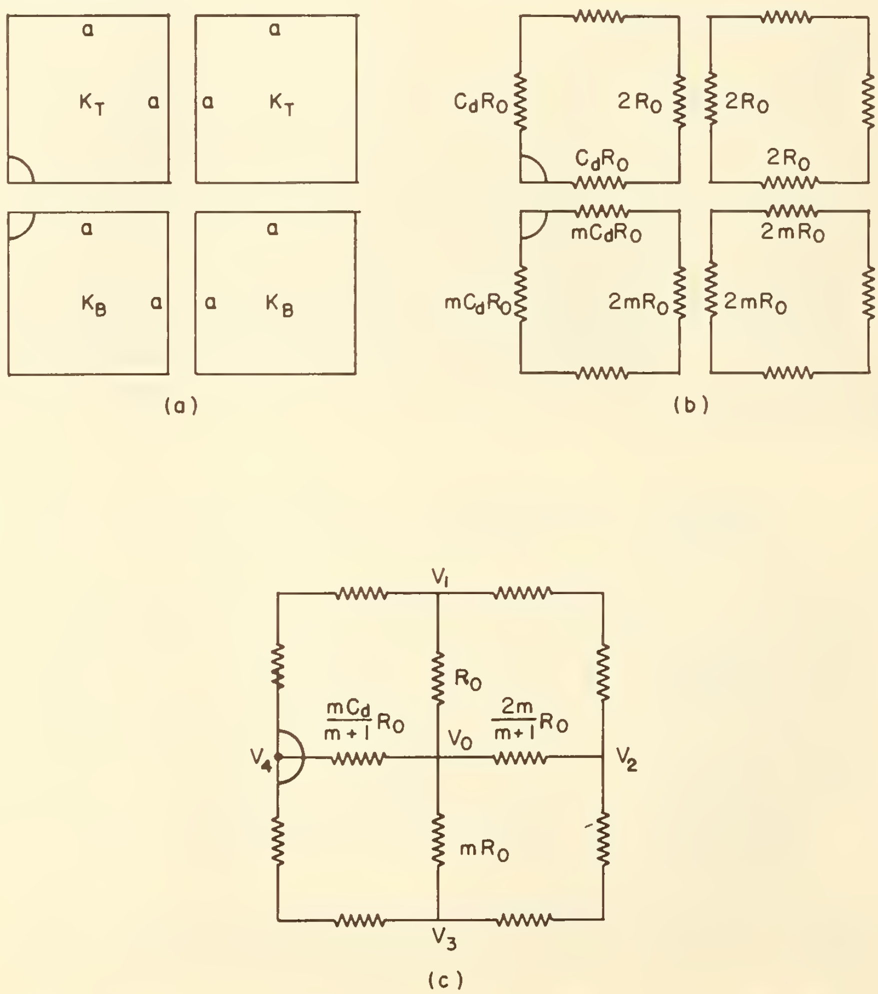



FIGURE 42.—Diagram showing building blocks of different hydraulic conductivity around the drain. The diagram is used to determine the potential $V_0$ at the nodes nearest the drain.

## III. SELECTING THE CHARACTERISTIC RESISTANCE OF THE NETWORK

There are two factors which must be considered in choosing the characteristic resistance $\underline{R}_o$ of the network: These factors are concerned with (1) the error introduced in voltage measurements and (2) the error resulting from contact resistance in the many jack and plug assemblies.

In voltage measurements a voltmeter is connected in parallel with the circuit. Since a voltmeter has an internal resistance of approximately 11 megohms, connecting the voltmeter to a circuit having a resistance $\underline{R}_o$ will yield a total resistance $\underline{R}$ as follows (see equation 6):

$$\underline{R} = \frac{11{,}000{,}000\underline{R}_o}{11{,}000{,}000+\underline{R}_o}$$

This will introduce an error in the measurement of $\underline{R}_o$ of $\frac{(\underline{R}-\underline{R}_o)}{\underline{R}_o}$ 100 percent, or $\frac{100\,\underline{R}_o}{11{,}000{,}000+\underline{R}_o}$ percent. The latter expression indicates that if $\underline{R}_o$ is large, one will have a high percentage of error. It is found that when a network of resistors is assembled to represent a soil profile, the overall resistance of the network is in the neighborhood of $\underline{R}_o$. It follows, therefore, that $\underline{R}_o$ should be chosen small to reduce errors in voltage measurement.

The second factors concerns errors resulting from wiring resistances and collective resistances in plug and jack assemblies. The overall resistance added by these sources is about 1 to 5 ohms. If the network resistance is around $\underline{R}_o$ ohms and the average wiring resistance is, say, 2.5 ohms, the percentage error due to wiring resistance is as follows:

$$\left(\frac{(\underline{R}_o + 2.5) - \underline{R}_o}{\underline{R}_o}\right) \times 100 = \frac{250}{\underline{R}_o} \text{ percent}$$

From this expression, one can see that if $\underline{R}_o$ is small, the percentage error will also be large.

With the two sources of error in mind, we can determine an optimum value for the characteristic resistance $\underline{R}_o$ of the network. This is done by first averaging the values of the logarithm of 11 megohms (internal resistance of the voltmeter) and of 2.5 ohms (wiring resistance). Then one takes the antilog of the average to obtain the value of the characteristic resistance $\underline{R}_o$. That is, log 11,000,000 $\approx$ 7 and log 2.5 $\approx$ 0.4 so that the antilog of $\frac{7 + 0.4}{2}$ = 5,000 ohms.

If an $\underline{R}_o$ of 5,000 ohms is chosen, the percentage error due to internal resistance of the voltmeter (upper limit) is 0.045 percent and the percentage error due to wiring resistance (lower limit) is 0.050 percent. Any deviation of $\underline{R}_o$ from 5,000 ohms will increase one and decrease the other. For example, if an $\underline{R}_o$ of 200 ohms is chosen, the error of the upper limit is 0.00018 percent and the error of the lower limit is increased to 1 percent.

## Using a Voltmeter to Measure Currents

One can use a digital voltmeter to measure current flow. This is done by attaching a standard resistor of 1,10, or 100 ohms in series with the circuit having resistance of $\underline{R}_o$ ohms and then using the voltmeter to measure the voltage across the standard resistor as shown in figure 43. The magnitude of $\underline{R}$ is not important, just as long as the voltage across it is the voltage desired. The current going through both the standard resistor and the resistance $\underline{R}$ will be the same since they are in series. By reading the voltage drop across a standard resistor, the current $\underline{I}$ can be determined as $\frac{\underline{V}}{\underline{R}\text{ standard}}$.

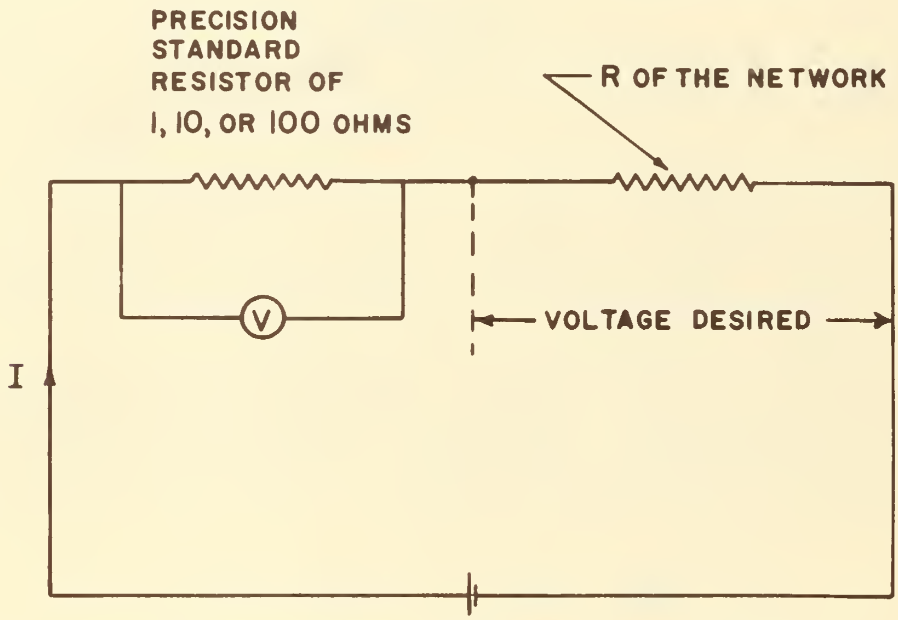



FIGURE 43.—Diagram showing the use of VTVM to measure current flow.

## Error in Setting Variable Resistors with a Wheatstone Bridge

In using the Wheatstone bridge to set a resistor to the desired value, one proceeds as follows: In two arms of the bridge place precision resistors (0.05 percent) of equal magnitude. Place resistance decade box of 0.05 percent accuracy covering a range from 0.1 ohm to 1 megohm in another arm of the bridge. Then, insert the resistor to be set into the remaining arm (fig. 44). To set the resistance, set the decade box to the desired value, and adjust the unknown resistor until the voltage $\underline{V}$ is zero. However, it is impractical to adjust the unknown resistor until the voltage $\Delta \underline{V}$ reads exactly zero. In such case, the desired resistance will have an error of $\Delta \underline{R}$. To find the percentage error, one proceeds as follows: Let the two fixed resistances and the decade box all have

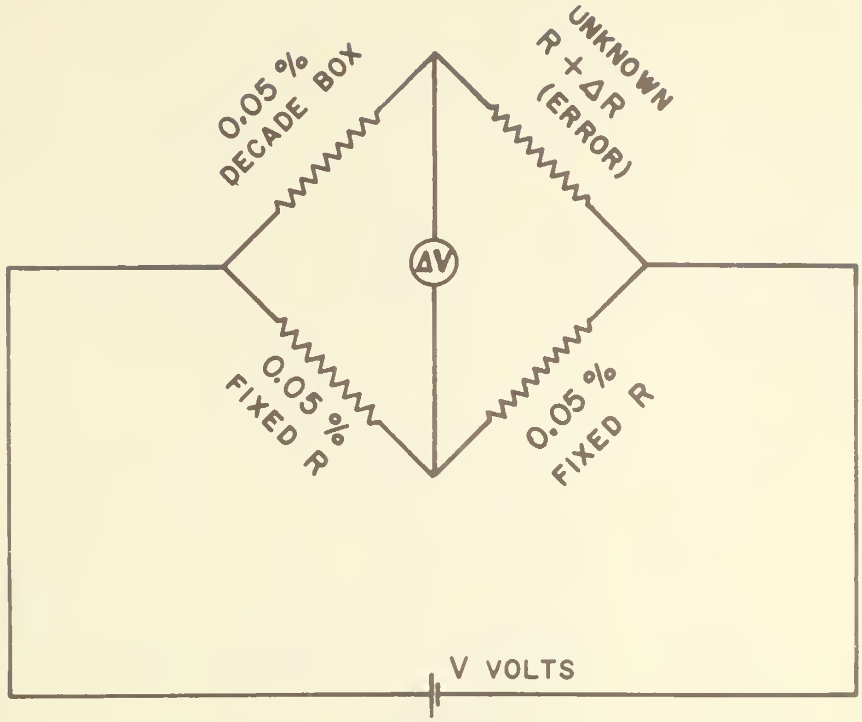



FIGURE 44.—Diagram showing the current of the Wheatstone bridge. The unknown resistor which is to be set to a certain value is put in the arm marked "unknown" and is adjusted until the voltage $\Delta \underline{V}$ approaches zero.

the same resistance $\underline{R}$. Apply a voltage $\underline{V}$ to the circuit. The amount of error in setting the resistance can be expressed as $\frac{\Delta \underline{R}}{\underline{R}} = \frac{4\,\Delta \underline{V}}{\underline{V}}$, so that the percentage error is $\frac{\Delta \underline{R} \times 100}{\underline{R}} = \frac{400\,\Delta \underline{V}}{\underline{V}}$.

In setting each resistance, apply a voltage $\underline{V}$ of not more than 30 volts to the bridge and keep the residual $\Delta \underline{V}$ below 0.025 volt. The percentage error in setting each resistance can be estimated as $\frac{400 \times 0.025}{30} = 0.33$ percent.

In the network system, these resistors are joined in mesh and the maximum errors should not exceed the amount $\frac{\Sigma \Delta \underline{R}}{\underline{n}}$ where $\underline{n}$ is the number of the resistors and $\Delta \underline{R}$ is the error of each resistor.

UNITED STATES DEPARTMENT OF AGRICULTURE

1862 1962

Growth Through Agricultural Progress